

Naugled.

Aerospace Dept

Air Craft

Structures

by

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## EQUILIBRIUM OF FORCES

**1.1. Equations of Equilibrium.** One of the first steps in the design of a machine or structure is the determination of the loads acting on each member. The loads acting on an airplane may occur in various landing or flight conditions. The loads may be produced by ground reactions on the wheels, by aerodynamic forces on the wings and other surfaces, or by forces exerted on the propeller. The loads are resisted by the weight or inertia of the various parts of the airplane. Several loading conditions must be considered, and each member must be designed for the combination of conditions which produces the highest stress in the member. For practically all members of the airplane structure the maximum loads occur when the airplane is in an accelerated flight or landing condition and the external loads are not in equilibrium. If, however, the inertia loads are also considered, they will form a system of forces which are in equilibrium with the external loads. In the design of any member it is necessary to find all the forces acting on the member, including inertia forces. Where these forces are in the same plane, as is often the case, the following equations of static equilibrium apply to any isolated portion of the structure:

$$\left. \begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma M &= 0 \end{aligned} \right\} \quad (1.1)$$

The terms  $\Sigma F_x$  and  $\Sigma F_y$  represent the summations of the components of forces along  $x$  and  $y$  axes, which may be taken in any two arbitrary directions. The term  $\Sigma M$  represents the sum of the moments of all forces about any arbitrarily chosen point in the plane. Each of these equations may be set up in an infinite number of ways for any problem, since the directions of the axes and the center of moments may be chosen arbitrarily. Only three independent equations exist for any free body, however, and only three unknown forces may be found from the equations. If, for example, an attempt is made to find four unknown forces by using the two force equations and moment equations about two points, the four equations cannot be solved because they are not independent, *i.e.*, one of the equations can be derived from the other three.



The following equations cannot be solved for the numerical values of the three unknowns because they are not independent.

$$\begin{aligned}x + y + z &= 3 \\x + y + 2z &= 4 \\2x + 2y + 3z &= 7\end{aligned}$$

The third equation may be obtained by adding the first two equations, and consequently does not represent an independent condition.

In the analysis of a structure containing several members it is necessary to draw a free-body diagram for each member, showing all the forces acting on that member. It is not possible to show these forces on a composite sketch of the entire structure, since equal and opposite forces act at all joints and an attempt to designate the correct direction of the force on each member will be confusing. In applying the equations of statics it is desirable to choose the axes and centers of moments so that only one unknown appears in each equation.

Many structural joints are made with a single bolt or pin. Such joints are assumed to have no resistance to rotation. The force at such a joint must pass through the center of the pin, as shown in Fig. 1.1, since the moment about the center of the pin must be zero. The force at the pin joint has two unknown quantities, the magnitude  $F$  and the direction  $\theta$ . It is usually more convenient to find the two unknown components,  $F_x$  and  $F_y$ , from which  $F$  and  $\theta$  can be found by the equations:

$$F = \sqrt{F_x^2 + F_y^2} \quad (1.2)$$

$$\tan \theta = \frac{F_y}{F_x} \quad (1.3)$$

The statics problem is considered as solved when the components  $F_x$  and  $F_y$  at each joint are obtained.

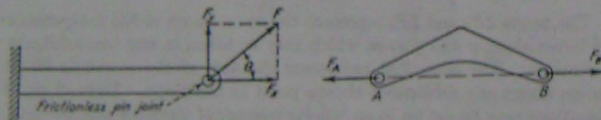


FIG. 1.1.

FIG. 1.2.

**1.2. Two-force Members.** When a structural member has forces acting at only two points, these forces must be equal and opposite, as shown in Fig. 1.2. Since moments about point  $A$  must be zero, the force  $F_B$  must pass through point  $A$ . Similarly the force  $F_A$  must pass through point  $B$  for moments about point  $B$  to be zero. From a summation of forces, the forces  $F_B$  and  $F_A$  must have equal magnitudes but

opposite directions. Two-force members are frequently used in aircraft and other structures, since simple tension or compression members are usually the lightest members for transmitting forces. Where possible, two-force members are straight, rather than curved as shown in Fig. 1.2. Structures made up entirely of two-force members are called trusses and are frequently used in fuselages, engine mounts, and other aircraft structures, as well as in bridge and building structures. Trusses represent an important special type of structure and will be treated in detail in the following articles.

Many structures contain some two-force members, as well as some members which resist more than two forces. These structures must first be examined carefully in order to determine which members are two-force members. Students frequently make the serious mistake of assuming that forces act in the direction of a member, when the member resists forces at three or more points. In the case of curved two-force members such as shown in Fig. 1.2, it is important to assume that the forces act along the line between pins, rather than along the axis of the members.

While Eqs. 1.1 are simple and well known, it is very important for a student to acquire proficiency in the application of these equations to various types of structures. A typical structure will be analyzed as an example problem.

**Example.** Find the forces acting at all joints of the structure shown in Fig. 1.3.

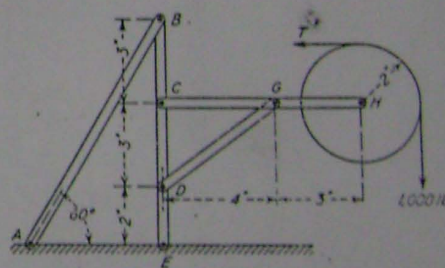


FIG. 1.3.

**Solution.** First draw free-body diagrams of all members, as shown in Fig. 1.4. Since  $AB$  and  $GD$  are two-force members, the forces in these members are along the line joining the pin joints of these members, and free-body diagrams of these members are not shown. The directions of forces are assumed, but care must be taken to show the forces at any joint in opposite directions on the two mem-

tem, that  $b$ ,  $C$ , is assumed to act to the right on the horizontal member, and therefore must act to the left on the vertical member. If forces are assumed in the wrong direction, the calculated magnitude will be negative.

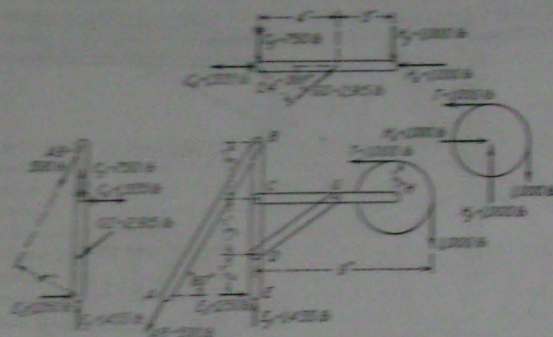


FIG. 1A

For the pulley,

$$\Sigma M_B = 2 \times 1,000 - 27 = 0$$

$$F = 1,000 \text{ lb}$$

$$\Sigma F_x = E_x - 1,000 = 0$$

$$E_x = 1,000 \text{ lb}$$

$$\Sigma F_y = E_y - 1,000 = 0$$

$$E_y = 1,000 \text{ lb}$$

Once the numerical values of these forces are obtained, they are shown on the free-body diagram. Subsequent equations contain the known numerical values rather than the algebraic symbols for the forces.

For member CDH,

$$\Sigma M_C = 1,000 \times 7 - 2,450 = 0$$

$$CD = 2,450 \text{ lb}$$

$$\Sigma F_x = C_x + 2,450 \cos 36.9^\circ - 1,000 = 0$$

$$C_x = -1,335 \text{ lb}$$

$$\Sigma F_y = C_y + 2,450 \sin 36.9^\circ - 1,000 = 0$$

$$C_y = -730 \text{ lb}$$

Since  $C_x$  and  $C_y$  are negative, the directions of the vectors on the free-body diagrams are changed. Such changes are made by crossing out the original arrows rather than by erasing, in order that the analysis may be checked conveniently by the original designer or by others. Extreme care must be observed in using the proper direction of known forces.

For member HJDE,

$$\Sigma M_D = 1,335 \times 5 - 2,915 \times 2 \cos 36.9^\circ - 4,552.5 = 0$$

$$AJ = 590 \text{ lb}$$

$$\Sigma F_x = E_x - 1,335 \cos 36.9^\circ + 1,335 - 590 \cos 63.7^\circ = 0$$

$$E_x = 1,250 \text{ lb}$$

$$\Sigma F_y = E_y - 1,335 \sin 36.9^\circ + 730 - 590 \sin 63.7^\circ = 0$$

$$E_y = 1,430 \text{ lb}$$

All forces have now been obtained without the use of the entire structure as a free body. The solution will be checked by using all three equations of equilibrium for the entire structure.

Check using entire structure as free body,

$$\Sigma F_x = 1,250 - 1,000 - 590 \cos 63.7^\circ = 0$$

$$\Sigma F_y = 1,430 - 1,000 - 590 \sin 63.7^\circ = 0$$

$$\Sigma M_G = 1,000 \times 9 - 1,000 \times 7 - 590 \times 4 = 0$$

This check should be made whenever possible in order to detect errors in computing moment arms or forces.

### PROBLEMS

1.1. A 1,000-lb airplane is in a steady glide with the flight path at an angle  $\theta$  below the horizontal. The drag force in the direction of the flight path is 730 lb. Find the lift force  $L$  normal to the flight path and the angle  $\theta$ .

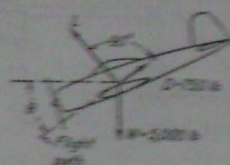


FIG. 1.1

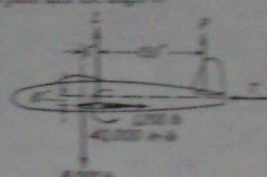


FIG. 1.2

1.2. A jet-propelled airplane in steady flight has forces acting as shown. Find the jet thrust  $T$ , lift  $L$ , and tail load  $P$ .

1.3. A wind-tunnel model of an airplane wing is suspended as shown. Find the loads in members  $BC$ , and  $EF$  if the forces at  $A$  are  $L = 43.5 \text{ lb}$ ,  $D = 2.62 \text{ lb}$ , and  $N = -21.6 \text{ lb}$ .

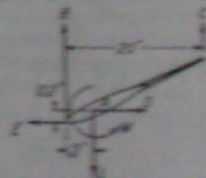


FIG. 1.3 and 1.4

1.4. For the model of Prob. 1.3 find the forces  $L$ ,  $D$ , and  $N$  at a point  $A$ . If the measured forces are  $B = 43.2 \text{ lb}$ ,  $C = 4.26 \text{ lb}$ , and  $E = 2.74 \text{ lb}$ .



## AIRCRAFT STRUCTURES

1.9. Find the horizontal and vertical components of the forces at all joints. The reaction at point  $B$  is vertical. Check results by using the three remaining equations of statics.

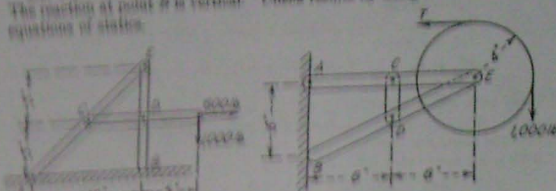


FIG. 1.9.

1.10. Find the horizontal and vertical components of the forces at all joints. The reaction at point  $B$  is horizontal. Check results by three equations.

1.11. Find the horizontal and vertical components of the forces at all joints. The reaction at point  $B$  is vertical. Check results by three equations.

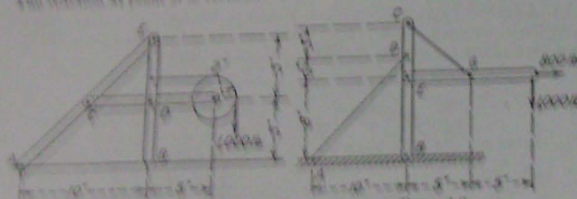


FIG. 1.10.

FIG. 1.11.

1.12. Find the forces at all joints of the structure. Check results by three equations.

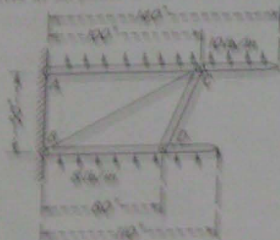


FIG. 1.12.

1.13. Find the forces on all members of the biplane structure shown. Check results by considering equilibrium of entire structure as a free body.

1.10. Find the forces at points  $A$  and  $B$  of the landing gear shown.  
1.11. Find the forces at points  $A$ ,  $B$ , and  $C$  of the structure of the braced-wing monoplane shown.  
1.12. Find the forces  $V$  and  $M$  at the cut cross section of the beam.

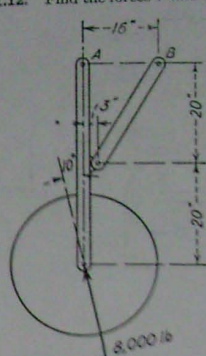


FIG. 1.10.

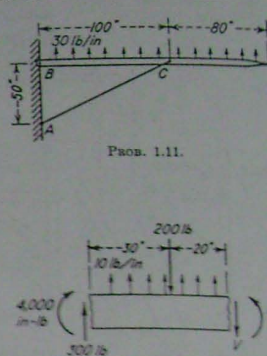


FIG. 1.11.

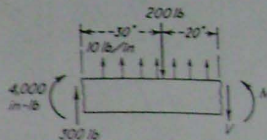


FIG. 1.12.

1.3. Truss Structures. A truss has been defined as a structure which is composed entirely of two-force members. In some cases the members have a single bolt or pin connection at each end, and the external loads are applied only at the pin joints. In other cases the members are welded or riveted at the ends, but are assumed to be pin-connected in the analysis because it has been found that such an analysis yields approximately the correct values for the forces in the members. The trusses considered in this chapter are assumed to be coplanar; the loads resisted by the truss and the axes of all of the truss members lie in the same plane.

Trusses may be classified as statically determinate and statically indeterminate. The forces in all of the members of a statically determinate truss may be obtained from the equations of statics. In a statically indeterminate truss, there are more unknown forces than the number of independent equations of statics, and the forces cannot be determined from the equations of statics. If a rigid structure is supported in such a manner that three nonparallel, nonconcurrent reaction components are developed, the three reaction forces may be obtained from the three equations of statics for the entire structure as a free body. If more than three support reactions are developed, the structure is statically indeterminate externally.



Trusses which have only three reaction force components, but which contain more members than required, are statically indeterminate internally. Trusses are normally formed of a series of triangular frames. The first triangle contains three members and three joints. Additional triangles are each formed by adding two members and one joint. The number of members  $m$  has the following relationship to the number of joints  $j$ .

$$m - 3 = 2(j - 3)$$

or

$$m = 2j - 3 \quad (1.4)$$

If a truss has one less member than the number specified by Eq. 1.4, it becomes a linkage or mechanism, with one degree of freedom. A linkage is not capable of resisting loads, and is classified as an unstable structure. If a truss has one more member than the number specified by Eq. 1.4, it is statically indeterminate internally.

If each pin joint of a truss is considered as a free body, the two statics equations  $\Sigma F_x = 0$  and  $\Sigma F_y = 0$  may be applied. The equation  $\Sigma M = 0$  does not apply, since all forces act through the pin, and the moments about the pin will be zero regardless of the magnitudes of the forces. Thus, for a truss with  $j$  joints, there are  $2j$  independent equations of statics. The equations for the equilibrium of the entire structure are not independent of the equations for the joints, since they can be derived from the equations of equilibrium for the joints. For example, the equation  $\Sigma F_x = 0$  for the entire structure may be obtained by adding all the equations  $\Sigma F_x = 0$  for the individual joints. The equations  $\Sigma F_y = 0$  and  $\Sigma M = 0$  for the entire truss may similarly be obtained from the equations for the joints. Equation 1.4 may therefore be derived in another manner by equating the number of unknown forces for  $m$  members and three reactions to the number of independent equations  $2j$ , or  $m + 3 = 2j$ .

It is necessary to apply Eq. 1.4 with care. The equation is applicable for the normal truss which contains a series of triangular frames and has three external reactions, such as the truss shown in Fig. 1.5(a). For other trusses it is necessary to determine by inspection that all parts of the structure are stable. The truss shown in Fig. 1.5(b) satisfies Eq. 1.4; yet the left panel is unstable, while the right panel has one more diagonal than is necessary. The truss shown in Fig. 1.5(c) is stable and statically determinate, even though it is not constructed entirely of triangular frames.

Some trusses may have more than three external reactions, and fewer members than are specified by Eq. 1.4, and be stable and statically

determinate. The number of reactions  $r$  may be substituted for the three in Eq. 1.4.

$$m = 2j - r \quad (1.4a)$$

The number of independent equations,  $2j$ , is therefore sufficient to obtain the  $m + r$  unknown forces for the members and the reactions. An example of a stable and statically determinate truss which has four reactions may be obtained from the truss of Fig. 1.5(a) by adding a horizontal reaction at the upper left-hand corner and removing the right-hand diagonal member.

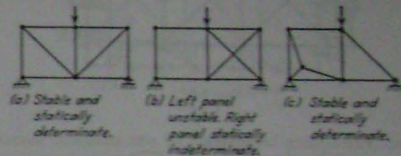


FIG. 1.5.

**1.4. Truss Analysis by Method of Joints.** In the analysis of a truss by the method of joints, the two equations of static equilibrium,  $\Sigma F_x = 0$  and  $\Sigma F_y = 0$ , are applied for each joint as a free body. Two unknown forces may be obtained for each joint. Since each member is a two-force member, it exerts equal and opposite forces on the joints at its ends. The joints of a truss must be analyzed in sequence by starting at a joint which has only two members with unknown forces. After finding the forces in these two members, an adjacent joint at the end of one of these members will have only two unknown forces. The joints are then analyzed in the proper sequence until all joints have been considered.

In most structures it is necessary to determine the three external reactions from the equations of equilibrium for the entire structure, in order to have only two unknown forces at each joint. These three equations are used in addition to the  $2j$  equations at the joints. Since there are only  $2j$  unknown forces, three of the equations are not necessary for finding the unknowns, but should always be used for checking the numerical work. The analysis of a truss by the method of joints will be illustrated by a numerical example.

**Example.** Find the loads in all the members of the truss shown in Fig. 1.6.

**Solution.** Draw a free-body diagram for the entire structure and for each joint, as shown in Fig. 1.7. Since all loads in the two-force members act along the members, it is possible to show all forces on a sketch of the truss, as shown in Fig. 1.7, if the forces are specified as acting on the joints. Care must be used in



the directions of the vectors, since at every point there is always a force acting on the member which is equal and opposite to the force acting on the joint. If a structure contains any members which are not two-force members, it is always necessary to make separate free-body sketches of these members, as shown in Fig. 1.4.

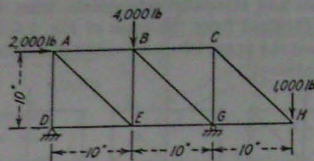


FIG. 1.6.

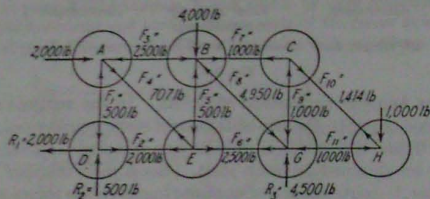


FIG. 1.7.

Considering the entire structure as a free body,

$$\begin{aligned}\Sigma M_D &= 2,000 \times 10 + 4,000 \times 10 + 1,000 \times 30 - 20R_3 = 0 \\ R_3 &= 4,500 \text{ lb} \\ \Sigma F_y &= R_2 - 4,000 - 1,000 + 4,500 = 0 \\ R_2 &= 500 \text{ lb} \\ \Sigma F_x &= 2,000 - R_1 = 0 \\ R_1 &= 2,000 \text{ lb}\end{aligned}$$

The directions of unknown forces are assumed, as in the previous example, and vectors changed on the sketch when they are found to be negative. Some engineers prefer to assume all members in tension, in which case negative signs designate compression. Joints must be selected in the proper order, so that there are only two unknowns at each joint.

Joint D:

$$\begin{aligned}\Sigma F_x &= F_2 - 2,000 = 0 \\ F_2 &= 2,000 \text{ lb} \\ \Sigma F_y &= 500 - F_1 = 0 \\ F_1 &= 500 \text{ lb}\end{aligned}$$

Joint A:

$$\begin{aligned}\Sigma F_y &= 500 - F_4 \sin 45^\circ = 0 \\ F_4 &= 707 \text{ lb} \\ \Sigma F_x &= 2,000 + 707 \cos 45^\circ - F_3 = 0 \\ F_3 &= 2,500 \text{ lb}\end{aligned}$$

Joint E:

$$\begin{aligned}\Sigma F_x &= F_6 - 2,000 - 707 \cos 45^\circ = 0 \\ F_6 &= 2,500 \text{ lb} \\ \Sigma F_y &= 707 \sin 45^\circ - F_5 = 0 \\ F_5 &= 500 \text{ lb}\end{aligned}$$

Joint B:

$$\begin{aligned}\Sigma F_y &= 500 - 4,000 + F_8 \sin 45^\circ = 0 \\ F_8 &= 4,950 \text{ lb} \\ \Sigma F_x &= 2,500 - 4,950 \cos 45^\circ + F_7 = 0 \\ F_7 &= 1,000 \text{ lb}\end{aligned}$$

Joint C:

$$\begin{aligned}\Sigma F_x &= F_{10} \cos 45^\circ - 1,000 = 0 \\ F_{10} &= 1,414 \text{ lb} \\ \Sigma F_y &= F_9 - 1,414 \sin 45^\circ = 0 \\ F_9 &= 1,000 \text{ lb}\end{aligned}$$

Joint G:

$$\begin{aligned}\Sigma F_x &= 4,950 \cos 45^\circ - 2,500 - F_{11} = 0 \\ F_{11} &= 1,000 \text{ lb}\end{aligned}$$

Check:

$$\Sigma F_x = 4,500 - 4,950 \cos 45^\circ - 1,000 = 0$$

Joint H:

$$\begin{aligned}\text{Check: } \Sigma F_y &= 1,414 \sin 45^\circ - 1,000 = 0 \\ \text{Check: } \Sigma F_x &= 1,000 - 1,414 \cos 45^\circ = 0\end{aligned}$$

Arrows acting toward a joint show that a member is in compression, and arrows acting away from a joint indicate tension.

**1.5. Truss Analysis by Method of Sections.** It is often desirable to find the forces in some of the members of a truss without analyzing the entire truss. The method of joints is usually cumbersome in this case, since the forces in all members to the left of any member must be obtained before finding the force in that particular member. An analysis by the method of sections will yield the force in any member by a single operation, without the necessity of finding the forces in the other members. Instead of considering the joints as free bodies, a cross section is taken through the truss, and the part of the truss on one side of the cross section is considered as a free body. The cross section is chosen so that it cuts the members for which the forces are desired and so that it preferably cuts only three members.



If the forces in members  $BC$ ,  $BG$ , and  $EG$  of the truss of Fig. 1.7 are desired, the free body will be as shown in Fig. 1.8. The three unknowns may be found from the three equations for static equilibrium.

$$\begin{aligned}\sum M_G &= 10F_1 - 10 \times 4,000 + 2,000 \times 10 + 500 \times 20 = 0 \\ F_1 &= 1,000 \text{ lb} \\ \sum F_y &= F_2 \sin 45^\circ - 4,000 + 500 = 0 \\ F_2 &= 4,950 \text{ lb} \\ \sum F_x &= F_3 + 1,000 - 4,950 \cos 45^\circ + 2,000 - 2,000 = 0 \\ F_3 &= 2,500 \text{ lb}\end{aligned}$$

These values check those obtained in the analysis by the method of joints.

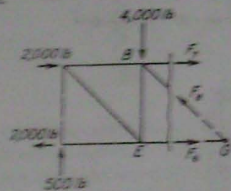


FIG. 1.8.

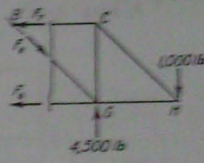


FIG. 1.9.

The portion of the truss to the right of the section through the members might have been taken as the free body, as shown in Fig. 1.9. The equations of equilibrium would be as follows:

$$\begin{aligned}\sum M_G &= 1,000 \times 10 - 10F_1 = 0 \\ F_1 &= 1,000 \text{ lb} \\ \sum F_y &= 4,500 - 1,000 - F_2 \sin 45^\circ = 0 \\ F_2 &= 4,950 \text{ lb} \\ \sum F_x &= 4,950 \cos 45^\circ - 1,000 - F_3 = 0 \\ F_3 &= 2,500 \text{ lb}\end{aligned}$$

It would also be possible to find the force  $F_3$  by taking moments about point  $B$ , thus eliminating the necessity of first finding the forces  $F_1$  and  $F_2$ .

**1.6. Truss Analysis—Graphic Method.** In the analysis of trusses by the method of joints, two unknown forces were obtained from the equations of equilibrium. It is also possible to find two unknown forces at each joint graphically by the use of the force polygon for the joint. The joints must be analyzed in the same sequence as used in the method of joints. In the graphic truss analysis, it is convenient to use Bow's notation, in which each space is designated by a letter and forces are

designated by the two letters corresponding to the spaces on each side of the force. The truss of Fig. 1.6 will be analyzed graphically, and the notation used will be as shown in Fig. 1.10. The capital letters designating the joints are usually omitted but are included here only for reference during the discussion.

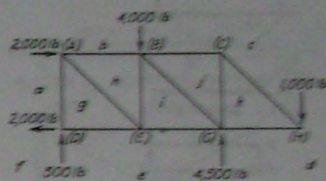


FIG. 1.10.

The external reactions can be determined graphically, but since it is usually more convenient to use algebraic methods the graphic solution will not be considered here. Using the external reactions found in Art. 1.4, a force polygon for the entire structure as a free body is shown in Fig. 1.11(a). While  $ab$  and  $fg$  are on the same horizontal line, and  $bc$  and  $de$  are on the same vertical line and will be shown that way in future work, they are shown displaced slightly for purposes of explanation. The notation shown is such that when the letters are read clockwise around the structure of Fig. 1.10,  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$ , and  $f$ , the directions of the forces in the force polygon will be  $a$  to  $b$ ,  $b$  to  $c$ ,  $c$  to  $d$ ,  $d$  to  $e$ , and  $e$  to  $f$ , with the polygon closing by the force  $fa$ .

Joint  $D$  is first considered as a free body and the known forces  $cd$  and  $de$  drawn to scale. The unknown forces  $ad$  and  $ge$  are then drawn in the proper directions from  $a$  and  $e$ , and the magnitudes are determined from the intersection  $d$ , as shown in Fig. 1.11(b). Joint  $A$  is next analyzed by drawing known forces  $ga$  and  $ab$  and finding  $ba$  and  $ha$  by the intersection at  $h$  in Fig. 1.11(c). Similarly, joints  $E$ ,  $B$ ,  $C$ , and  $G$  are analyzed as shown in Figs. 1.11(d) to (g). All forces have now been determined without considering joint  $H$ . The force polygon for joint  $H$ , shown in Fig. 1.11(h), is used for checking results, as in the algebraic solution.

A study of the force polygons shows that each force appears in two polygons. Force  $ge$  is a force to the right on joint  $D$ , and  $eg$  is a force to the left on joint  $E$ , but points  $e$  and  $g$  have the same relative position in both diagrams. All the polygons can be combined into one stress diagram as shown in Fig. 1.11(i). Arrows are omitted from the stress diagram, since there would always be forces in both directions and the arrows would have no significance. To determine the direction of the



forces at any joint, the letters are read clockwise around the joint. Thus, at joint *D* in Fig. 1.10, the letters *ge* are read, which are seen in Fig. 1.11(f) to represent a force to the right on joint *D*. Proceeding

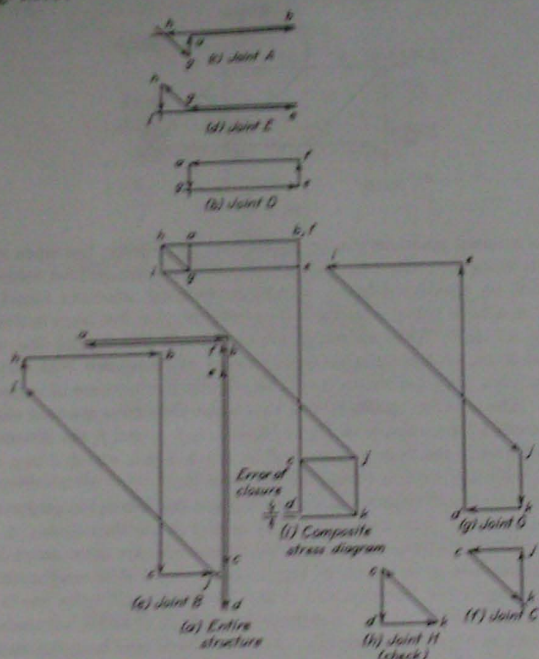


FIG. 1.11.

clockwise around joint *E*, the letters *eg* are read, which represent a force to the left in the stress diagram. The member is therefore a tension member exerting a force to the left on joint *E*.

**Example.** Construct a stress diagram for the steel-tube fuselage truss shown in Fig. 1.12. The structure is stable and statically determinate although space *c* is not triangular. A lighter structure would be obtained if a single diagonal were used in place of members *ac*, *cd*, and *de*, but this would not permit

enough space for a side door in the fuselage. The type of framing shown is often used to permit openings in the truss for access purposes.

**Solution.** The reactions  $R_1$  and  $R_2$  are first obtained algebraically. The stress diagram is then constructed as shown in Fig. 1.12(b). The forces in all members are then obtained by scaling the lengths from the stress diagram. The

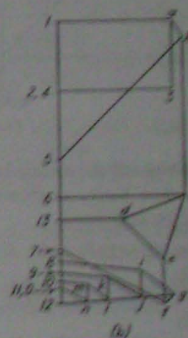
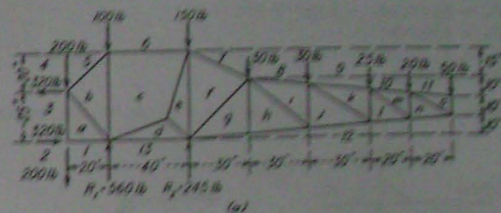


FIG. 1.12.

directions of the forces are obtained from the stress diagram in the same manner as in the previous example. For member 5-4, the line in the stress diagram is from left to right, indicating a tension force acting to the right on the joint to the left of the member. Reading clockwise around the joint to the right of the member, the force is 4-8, which in the stress diagram is a force to the left. This checks the conclusion that the member is in tension.

An algebraic solution of problems such as the fuselage truss may become rather tedious, since each of the inclined members is at a different angle. The graphic solution has many advantages for a problem of this type, because it is much easier to draw the truss to scale and to project lines parallel to the members than to calculate the angles and forces for the members.



1.7. Trusses Containing Members in Bending. Many structures are made up largely of two-force members but contain some members which are loaded laterally, as shown in Fig. 1.13. These structures are usually classed as trusses, since the analysis is similar to that used for trusses. The horizontal members of the truss shown in Fig. 1.13 are

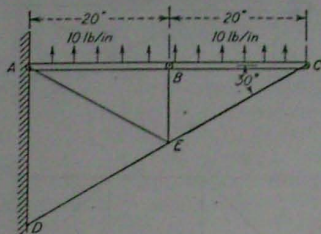


FIG. 1.13.

not two-force members, and separate free-body diagrams for these members, as shown in Figs. 1.14(a) and (b), are required. Since each of these members has four unknown reactions, the equations of statics are not sufficient for finding all four forces. It is possible to find the vertical forces  $A_y = B_{y1} = B_{y2} = C_y = 100$  lb and to obtain the relations  $A_x = B_{x1}$  and  $B_{x2} = C_x$  from the equilibrium equations for the horizontal members.

When the forces obtained from the horizontal members are applied to the remaining structure as a free body, as shown in Fig. 1.14(c), it is apparent that the remaining structure may be analyzed by the same methods that were used in the previous truss problems. The loads obtained by such an analysis are shown in Fig. 1.14(d). All members except the horizontal members may now be designed as simple tension or compression members. The horizontal members must be designed for bending moments combined with the compression load of 173.2 lb.

In the trusses previously analyzed, the members themselves have been assumed to be weightless. The effects of the weight of the members may be considered by the method used in the preceding example. It will be noticed that the correct axial loads in the truss members may be obtained if half the weight of the member is applied at each of the panel points at the ends of the member. The bending stresses in the member resulting from the weight of the member must be computed separately and combined with the axial stresses in the member.

Many trusses used in aircraft and other structures do not have frictionless pins at each end. Aircraft trusses are usually made of steel tubes with welded ends. While the truss members are not free to rotate at the ends in the same manner as frictionless pin joints, the members are flexible in bending when compared with the flexibility of the entire

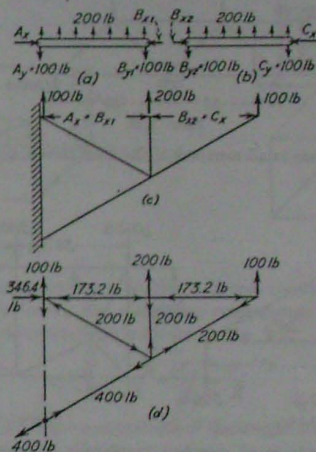


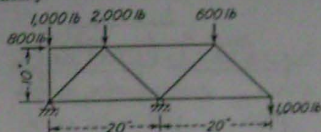
FIG. 1.14.

truss. It is customary and reasonably accurate to assume such trusses as pin-ended for analysis. This same assumption is also used for heavy bridge and building truss members, although bending stresses resulting from truss deflections are occasionally calculated for some bridge members. It can be shown that the ultimate strength of tension members can be predicted more accurately when these secondary bending stresses are neglected, since the bending stresses are relieved when the material yields slightly. Compression members are much stronger when the ends are rigidly welded than when the ends are pinned. The centroidal axes of all truss members should meet in one point at a joint. When clearance or manufacturing requirements do not permit members to meet at a point, the members must be designed for bending stresses produced by the eccentric loading.



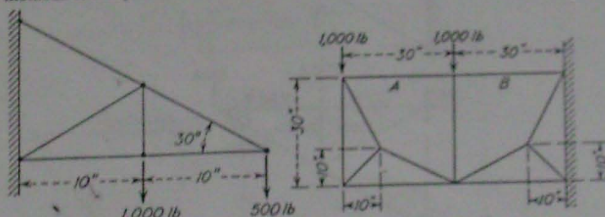
## PROBLEMS

1.13. Find the loads in all members of the truss by each of the following methods: (a) algebraically by method of joints; (b) algebraically by method of sections; (c) graphically.



PROB. 1.13.

1.14. Find the loads in all members of the truss shown, by each of the three methods of analysis.

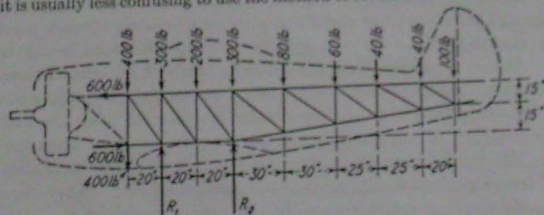


PROB. 1.14.

PROB. 1.15.

1.15. Find the loads in all members graphically and algebraically by the method of joints.

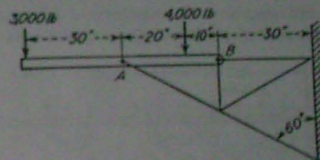
*Hint:* First obtain the loads in members *A* and *B* by the method of sections. It is possible to solve this truss graphically by the use of "phantom" members, but it is usually less confusing to use the method of sections for members *A* and *B*.



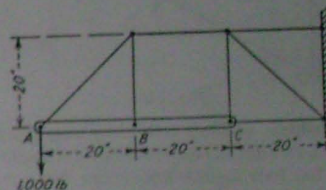
PROB. 1.16.

1.16. Find the loads in all members by graphic methods. The reactions  $R_1$  and  $R_2$  are obtained algebraically. Show numerical values of all forces on a diagram of the structure, indicating the directions by arrows.

1.17. The structure shown represents an engine mount for a V-type engine. Find the reactions on member *AB* and the forces in other members of the structure.



PROB. 1.17.



PROB. 1.18.

1.18. All members of the structure shown are two-force members, except member *ABC*. Find the reactions on member *ABC* and the loads in other members of the structure.



## CHAPTER 2

### SPACE STRUCTURES

**2.1. Equations of Equilibrium.** Most structures must be designed to resist loads acting in more than one plane. Consequently, the structures are actually space structures, although in many cases the loads in each plane may be considered independently and the structures analyzed by the methods of analysis for coplanar structures. When it is necessary to consider simultaneously the forces acting in more than one plane, the methods of analysis are no more difficult, but it is often more difficult to visualize the space geometry of the structure. In an analysis of space structures it is desirable to draw several views of the structure, with the forces shown in all views.

The equilibrium of any free body in space is defined by six equations:

$$\left. \begin{aligned} \sum F_x &= 0 & \sum M_x &= 0 \\ \sum F_y &= 0 & \sum M_y &= 0 \\ \sum F_z &= 0 & \sum M_z &= 0 \end{aligned} \right\} \quad (2.1)$$

The first three equations represent the summation of force components along three nonparallel axes, which may be chosen arbitrarily. The second three equations represent the summation of moments about three nonparallel axes. For a free body in space, it is possible to find six unknown forces from the equations of statics. Six reaction components are necessary for the stability of a space structure.

The components of a force  $R$  in space along three mutually perpendicular axes,  $x$ ,  $y$ , and  $z$ , may be obtained from the following equations:

$$\left. \begin{aligned} F_x &= R \cos \alpha \\ F_y &= R \cos \beta \\ F_z &= R \cos \gamma \end{aligned} \right\} \quad (2.2)$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are the angles between the force and the  $x$ ,  $y$ , and  $z$  axes, respectively, as shown in Fig. 2.1. When the three components are known, the resultant may be obtained from the following equation:

$$R = \sqrt{F_x^2 + F_y^2 + F_z^2} \quad (2.3)$$

Two-force members are frequently used in space structures as well as in coplanar structures. Theoretically, such members would require end

conditions similar to ball and socket joints in order to eliminate bending in any direction, but the rigidity of the usual types of trusses is such that bending of the members may be neglected, as in coplanar structures. A two-force member in space resists only tension or compression, and the force in such a member represents only one unknown quantity to be

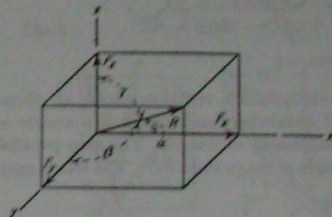


FIG. 2.1.

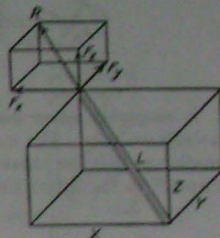


FIG. 2.2.

obtained from the equations of statics. If the resultant force or one of its components is found, the remaining components may be obtained from the geometric relationships shown in Fig. 2.2:

$$\frac{R}{L} = \frac{F_x}{X} = \frac{F_y}{Y} = \frac{F_z}{Z} \quad (2.4)$$

where  $X$ ,  $Y$ , and  $Z$  are the components of the length  $L$  along the mutually perpendicular reference axes.

**2.2. Moments and Couples.** The moment of a force about a line is obtained by projecting the force to a plane perpendicular to the line and finding the moment of the component of the force in that plane. The force  $P$  in Fig. 2.3, has components  $P_1$ , parallel to the axis of moments, and  $P_2$ , in a plane perpendicular to the axis of moments. The moment of the force about the line  $OO$  is  $P_2d$ , since the component  $P_1$  has no moment about the line. It will be noticed that a force has no moment about any line that is in the same plane as the force.

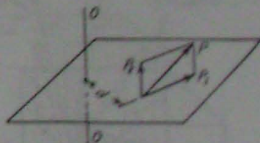


FIG. 2.3.

A couple consists of two equal parallel forces acting in opposite directions. It is found by taking moments about any arbitrary point in the plane of a couple that the moment of the couple is the same about all points in the plane. Thus the effect of the couple is not changed by



moving the couple in the plane. The couples shown in Fig. 2.4 are equivalent to each other, since they all have a clockwise moment  $Pd$  about any point in the plane. One quantity is therefore sufficient to define a couple in a plane.

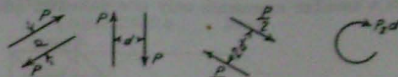


FIG. 2.4.

A couple in space may tend to produce rotation about all three coordinate axes. The method of obtaining components of a couple is similar to that for obtaining components of forces. The two parallel forces  $P$ , in Fig. 2.5(a), form a couple of magnitude  $Pd$ . The forces may

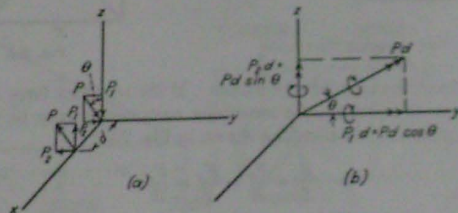


FIG. 2.5.

be resolved into components  $P_2$  and  $P_1$  parallel to the  $y$  and  $z$  axes. The components form couples  $P_2d$  and  $P_1d$  about the  $y$  and  $z$  axes which have magnitudes  $Pd \cos \theta$  and  $Pd \sin \theta$ , respectively. These components are

seen from Fig. 2.5(b) to be the same as would be obtained by representing the couple  $Pd$  by a single vector perpendicular to the plane of the couple and projecting this vector on axes perpendicular to the planes of the desired component couples. Couples will be represented by double-arrow vectors in order to distinguish them from force vectors, and the direction of a couple vector will be obtained by the left-hand rule; the arrows will be

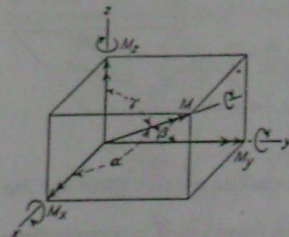


FIG. 2.6.

in the direction of the left thumb when the fingers are curved in the direction of rotation. While the couple shown has no moment about the  $x$  axis, the method of resolving a couple vector into three compo-

nents is identical with the method of resolving a force vector into three components. Considering Fig. 2.6,

$$\begin{aligned} M_x &= M \cos \alpha \\ M_y &= M \cos \beta \\ M_z &= M \cos \gamma \end{aligned} \quad (2.5)$$

Since a couple can be moved to any point in a plane, the couple vector, unlike a force vector, may be moved laterally to any point. A couple vector, like a force vector, may be moved along its line of action, since a couple may be moved from one plane to any parallel plane.

### 2.3. Analyses of Typical Space Structures

**Example 1.** Find the loads in the two-force members  $OA$ ,  $OB$ , and  $OC$  of the structure shown in Fig. 2.7.

**Solution.** Since all forces in the structure act through point  $O$ , the moments about any axis through point  $O$  will be zero regardless of the magnitudes of the forces. Consequently only the three equations of statics for the summation of forces along the axes are used to find the three unknowns. In Fig. 2.7, the

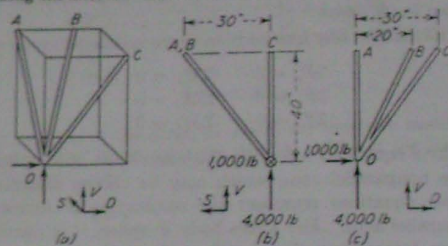


FIG. 2.7.

three mutually perpendicular axes are designated as the  $V$ ,  $D$ , and  $S$  axes. This notation is customary in the analysis of airplane landing gears to represent the vertical, drag, and side components of forces. The direction cosines of a member are obtained as  $V/L$ ,  $D/L$ , and  $S/L$ , where  $V$ ,  $D$ , and  $S$  represent the projected length of the member along the reference axes and  $L$  represents the true length of the member. The direction cosines are obtained in Table 2.1.

TABLE 2.1

| Member | $V$ | $D$ | $S$ | $L = \sqrt{V^2 + D^2 + S^2}$ | $\frac{V}{L}$ | $\frac{D}{L}$ | $\frac{S}{L}$ |
|--------|-----|-----|-----|------------------------------|---------------|---------------|---------------|
| A      | 40  | 0   | 30  | 50                           | 0.8           | 0             | 0.6           |
| B      | 40  | 20  | 30  | 53.9                         | 0.743         | 0.371         | 0.557         |
| C      | 40  | 30  | 0   | 50                           | 0.8           | 0.6           | 0             |



All members are assumed to be in tension, and summations of forces along the axes are as follows:

$$\Sigma F_x = 0.8A + 0.743B + 0.8C + 4,000 = 0$$

$$\Sigma F_y = 0.371B + 0.6C + 1,000 = 0$$

$$\Sigma F_z = 0.6A + 0.557B = 0$$

Solving these equations simultaneously, the forces are obtained as follows:

$$A = -5,000 \text{ lb} \quad \text{compression}$$

$$B = +5,390 \text{ lb} \quad \text{tension}$$

$$C = -5,000 \text{ lb} \quad \text{compression}$$

**Alternate Solution.** The above solution requires that three simultaneous equations be solved for the three unknowns. In coplanar structures it is usually convenient to set up the equations of statics so that each equation contains only one unknown. In space structures this is not always convenient, because of the more complicated geometry. If a summation of forces perpendicular to plane  $AOB$  is taken, only the unknown  $C$  will appear in the equation, but it is more difficult to solve for the necessary angles than to solve the simultaneous equations. A better method of obtaining equations which have only one unknown in each equation is to use the moment equations. Since the forces are concurrent, only three independent equations of statics exist, but three moment equations may be used, or any combination of moment and force equations. The unknowns will be taken as shown in Fig. 2.8.

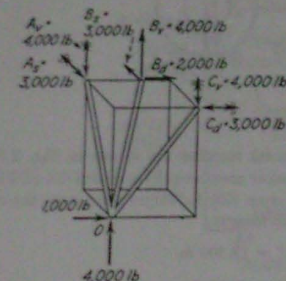


FIG. 2.8.

When one component of a force is found from the equations of statics, the other components are obtained by Eqs. 2.4. First taking moments about axis  $AB$ ,

$$\Sigma M_{AB} = 4,000 \times 30 + 30C_x = 0$$

$$C_x = -4,000 \text{ lb}$$

From Eqs. 2.4,

$$\frac{-4,000}{40} = \frac{C_x}{30} = \frac{C}{50}$$

$$C_x = -3,000 \text{ lb} \quad C = -5,000 \text{ lb}$$

These forces are now shown in the correct direction on Fig. 2.8. Force  $B$  may now be found from a summation of forces along the  $D$  axis:

$$\Sigma F_d = B_d - 3,000 + 1,000 = 0$$

$$B_d = 2,000 \text{ lb}$$

From Eqs. 2.4,

$$\frac{2,000}{20} = \frac{B_x}{40} = \frac{B_y}{30} = \frac{B_z}{53.9}$$

$$B_x = 4,000 \text{ lb} \quad B_y = 3,000 \text{ lb} \quad B_z = 5,390 \text{ lb}$$

The unknown force at  $A$  may now be found from any of the four remaining equations of statics:

$$\Sigma F_x = A_x + 3,000 = 0$$

$$A_x = -3,000 \text{ lb}$$

From Eqs. 2.4,

$$\frac{-3,000}{30} = \frac{A_y}{40} = \frac{A_z}{50}$$

$$A_y = -4,000 \text{ lb} \quad A_z = -5,000 \text{ lb}$$

$$\text{Check: } \Sigma F_z = 4,000 - 4,000 + 4,000 - 4,000 = 0$$

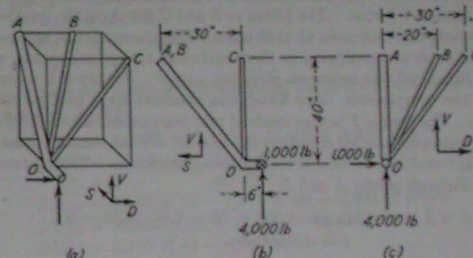


FIG. 2.9.

**Example 2.** Find the forces at points  $A$ ,  $B$ , and  $C$  for the landing gear of Fig. 2.9. Members  $OB$  and  $OC$  are two-force members. Member  $OA$  resists bending and torsion, but point  $A$  is hinged by a universal joint so that the member can carry torsion but no bending in any direction at this point.

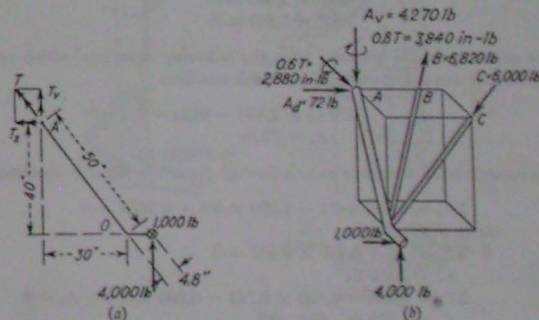


FIG. 2.10.

**Solution.** First consider the components of the torsional couple at point  $A$ . The resultant couple vector  $T$ , shown in Fig. 2.10(a), must be along the member,



and it has components  $T_v$  about a vertical axis and  $T_s$  about a side axis. By proportion,

$$\frac{T_v}{40} = \frac{T_s}{30} = \frac{T}{50}$$

or

$$T_v = 0.8T$$

and

$$T_s = 0.6T$$

In the free-body diagram for the entire structure, shown in Fig. 2.10(b), there are six unknown forces. The forces at  $B$  and  $C$  act along the member; therefore there is only one unknown at each point. The force at  $A$  is unknown in direction and must be considered as three unknown force components, or as an unknown force and two unknown direction angles. Usually it is more convenient to find the components, after which the resultant force may be obtained from Eq. 2.3. The couple  $T$  is also resolved into components about the  $S$  and  $V$  axes. The direction cosines for members  $OB$  and  $OC$  are the same as those obtained in Example 1 and will be used in the following equations. Taking moments about an axis through points  $A$  and  $B$ ,

$$\begin{aligned}\Sigma M_{AB} &= 4,000 \times 36 - 0.8C \times 30 = 0 \\ C &= 6,000 \text{ lb}\end{aligned}$$

All the unknown forces and the 4,000-lb load act through member  $OA$ . The torsional couple  $T$  may be found by taking moments about line  $OA$ . The 1,000-lb drag load has a moment arm of 4.8 in., as shown in Fig. 2.10(a).

$$\begin{aligned}\Sigma M_{AO} &= 1,000 \times 4.8 - T = 0 \\ T &= 4,800 \text{ in-lb} \\ 0.6T &= 2,880 \text{ in-lb} \\ 0.8T &= 3,840 \text{ in-lb}\end{aligned}$$

The other forces are obtained from the following equations, which are chosen so that only one unknown appears in each equation.

$$\begin{aligned}\Sigma M_{OS} &= 2,880 - 40A_s = 0 \\ A_s &= 72 \text{ lb}\end{aligned}$$

The subscripts  $OS$  designate an axis through point  $O$  in the side direction.

$$\begin{aligned}\Sigma F_v &= 1,000 + 72 - 6,000 \times 0.6 + 0.371B = 0 \\ B &= 6,820 \text{ lb}\end{aligned}$$

$$\begin{aligned}\Sigma F_s &= A_s - 6,820 \times 0.557 = 0 \\ A_s &= 3,800 \text{ lb}\end{aligned}$$

$$\begin{aligned}\Sigma F_v &= 4,000 + 6,820 \times 0.743 - 6,000 \times 0.8 - A_v = 0 \\ A_v &= 4,270 \text{ lb}\end{aligned}$$

$$\text{Check: } \Sigma M_{AV} = -1,000 \times 36 + 6,000 \times 0.6 \times 30 - 6,820 \times 0.557 \times 20 + 3,840 = 0$$

**Example 2.** Find the forces acting on all members of the landing gear shown in Fig. 2.11.

## SPACE STRUCTURES

**Solution.** For convenience, the reference axes,  $V$ ,  $D$ , and  $S$ , will be taken as shown in Fig. 2.11, with the  $V$  axis parallel to the oleo strut. Free-body dia-

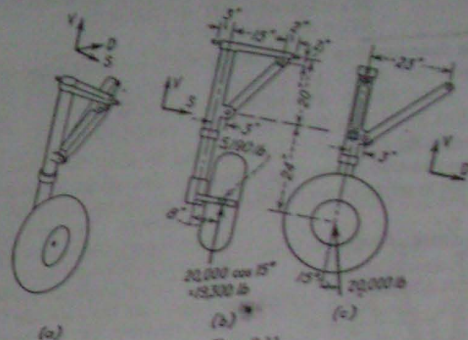


FIG. 2.11.

grams for the oleo strut and the horizontal member are shown in Fig. 2.12. Forces perpendicular to the plane of the paper are shown by a circled dot ( $\odot$ ) for forces toward the observer and a circled cross ( $\otimes$ ) for forces away from the observer. The  $V$  component of the 20,000-lb force is

$$20,000 \cos 15^\circ = 19,300 \text{ lb}$$

The  $D$  component is

$$20,000 \sin 15^\circ = 5,190 \text{ lb}$$

The angle of the side brace member  $CG$  with the  $V$  axis is

$$\tan^{-1} \frac{5,190}{19,300} = 33.7^\circ$$

The  $V$  and  $S$  components of the force in member  $CG$  are

$$\begin{aligned}CG \cos 33.7^\circ &= 0.832CG \\ CG \sin 33.7^\circ &= 0.535CG\end{aligned}$$

The drag-brace member  $BH$  is at an angle of  $45^\circ$  with the  $V$  axis, and the components of the force in this member along the  $V$  and  $D$  axes are

$$\begin{aligned}BH \cos 45^\circ &= 0.707BH \\ BH \sin 45^\circ &= 0.707BH\end{aligned}$$

The six unknown forces acting on the oleo strut are now obtained from the following equations:

$$\begin{aligned}\Sigma M_{AV} &= 5,190 \times 8 - T_v = 0 \\ T_v &= 41,720 \text{ in-lb}\end{aligned}$$

$$\begin{aligned}\Sigma M_{SS} &= 5,190 \times 44 - 0.707BH \times 20 - 0.707BH \times 3 = 0 \\ BH &= 14,030 \text{ lb}\end{aligned}$$

$$0.70207 = 0.800 \text{ lb}$$

$$\Sigma M_{ay} = 0.55507 \times 20 + 0.55207 \times 3 = 10.800 \times 3 = 0$$

$$C_2 = 11.300 \text{ lb}$$

$$0.55507 = 0.800 \text{ lb}$$

$$0.55207 = 0.600 \text{ lb}$$

$$\Sigma F_x = 10.800 + 0.800 - 0.600 - B_1 = 0$$

$$B_1 = 10.700 \text{ lb}$$

$$\Sigma F_z = B_2 - 0.800 = 0$$

$$B_2 = 0.800 \text{ lb}$$

$$\Sigma F_y = -5.100 + 0.600 - B_3 = 0$$

$$B_3 = 4.700 \text{ lb}$$

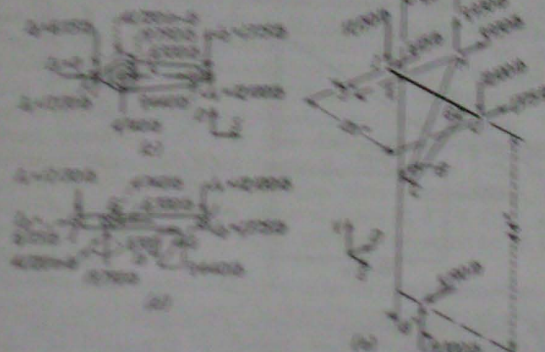
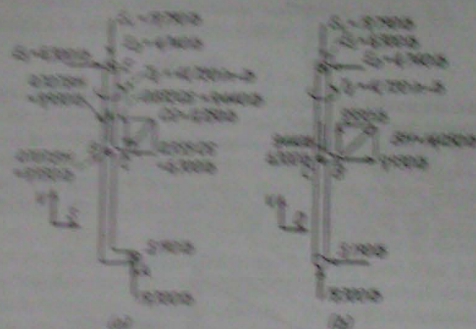


FIG. 2.12

The horizontal member  $BD$  will now be considered as a free body. The free bodies obtained above are applied to this member as shown in Figs. 2.12(c) and (d), and the free unknown reactions are obtained as follows:

$$\Sigma F_x = B_1 = 0$$

$$\Sigma M_{Dy} = 10.700 \times 3 + 0.600 \times 18 + 0.800 \times 2 - 20C_2 = 0$$

$$C_2 = 12.100 \text{ lb}$$

$$\Sigma F_z = 10.700 + 0.600 - 12.100 - B_2 = 0$$

$$B_2 = 12.100 \text{ lb}$$

$$\Sigma M_{Dz} = 40.700 - 4.700 \times 3 + 20C_3 = 0$$

$$C_3 = -1.375 \text{ lb}$$

$$\Sigma F_y = 4.700 + 1.375 - B_3 = 0$$

$$B_3 = 6.113 \text{ lb}$$

The reactions are now checked by considering the entire structure as a free body, as shown in Fig. 2.12(e).

$$\Sigma F_x = 10.800 - 12.100 - 12.100 + 0.600 = 0$$

$$\Sigma F_z = -5.100 + 1.375 - 6.113 + 0.800 = 0$$

$$\Sigma F_y = 0$$

$$\Sigma M_{Dy} = 0.100 \times 11 - 1.375 \times 20 - 0.600 \times 3 = 0$$

$$\Sigma M_{Dz} = 10.800 \times 11 - 12.100 \times 20 + 0.800 \times 3 = 0$$

$$\Sigma M_{Dx} = 5.100 \times 44 - 0.800 \times 25 = 0$$

**Example 4.** The conventional landing-gear shock absorber contains two telescoping tubes, as shown in Fig. 2.13(a). As the shock absorber is compressed, oil is forced through an orifice into an airtight chamber, and the energy

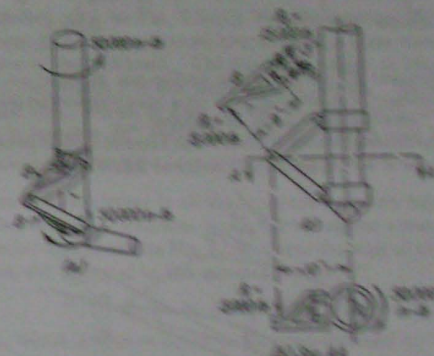


FIG. 2.13

from the landing impact is absorbed by the oil and air. This shock-absorber mechanism, called an oleo strut, transmits bending moment through the two telescoping tubes. The tubes, however, are free to rotate with respect to each



other, and additional structural members, called *torque links*, are required to resist torsion. The loads on the torque links *B* of the landing gear shown in Fig. 2.13 are desired.

**Solution.** Considering the lower tube and lower torque link as a free body, the forces are as shown in Fig. 2.13(b). Taking moments about the tube center line,

$$\Sigma M = 50,000 - 10R_1 = 0$$

$$R_1 = 5,000 \text{ lb}$$

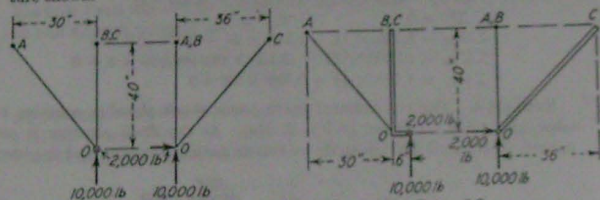
Now considering one of the torque links as a free body, as shown in Fig. 2.13(c), and taking moments about an axis perpendicular to the plane of the torque link,

$$\Sigma M = 3R_2 - 5,000 \times 9 = 0$$

$$R_2 = 15,000 \text{ lb}$$

## PROBLEMS

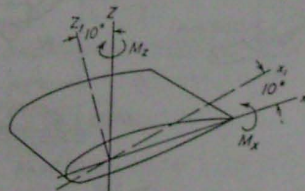
2.1. Find the forces in the two-force members *AO*, *BO*, and *CO* of the structure shown.



PROB. 2.1.

2.2. Members *AO* and *BO* of this landing-gear structure are two-force members. Member *CO* is attached at *C* by a joint which transmits torsion but no bending. Find the forces acting at all joints.

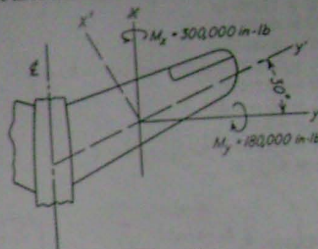
2.3. The bending moments about *x* and *z* axes in a plane perpendicular to the spanwise axis of a wing are 400,000 in-lb and 100,000 in-lb as shown. Find the



PROB. 2.3.

bending moments about *x*<sub>1</sub> and *z*<sub>1</sub> axes which are in the same plane but rotated 10° counterclockwise.

2.4. The main beam of the wing shown has a sweepback angle of 30°. The moments of 300,000 in-lb and 180,000 in-lb are first computed about axes *x* and *y*



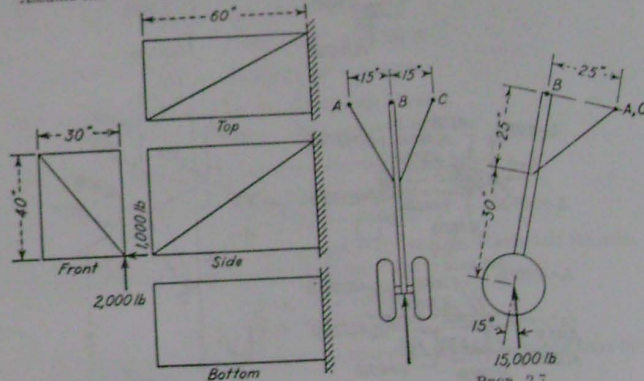
PROB. 2.4.

which are parallel and perpendicular to the center line of the airplane. Find the moments about axes *x'* and *y'*.

2.5. Find the forces in all members of the structure shown.

2.6. Omit the front diagonal member of the structure in Prob. 2.5, and add a diagonal member in the bottom plane. Find the forces in all members.

2.7. Find the forces acting on all members of the nose-wheel structure shown. Assume the *V* axis parallel to the oleo strut.



PROB. 2.5.

PROB. 2.7.

2.8. Analyze the landing-gear structure of Example 3, Art. 2.3, for a 15,000-lb load up parallel to the *V* axis and a 5,000-lb load aft parallel to the *D* axis. The loads are applied at the same point of the axle as the load in the example problem.



**2.4. Torsion of Space Frameworks.** Space structures composed of two-force members may be loaded in various ways. For a structure such as a truss-type airplane fuselage it is often convenient to consider separately conditions such as vertical bending, side bending, and torsion, and then superimpose the results obtained from each analysis. In the analysis for a symmetrical loading condition such as vertical bending it is usually possible to consider each side of the structure as a coplanar truss and to determine the loads in all members from the equilibrium of coplanar forces. It is only in the analysis for torsional loads that the methods for space structures need be used.

A proportionality between the components of length and the components of force in a two-force member was shown in Eqs. 2.4. A tension coefficient  $\mu$  is often used and is defined as follows:

$$\mu = \frac{R}{L} = \frac{F_x}{X} = \frac{F_y}{Y} = \frac{F_z}{Z} \quad (2.6)$$

The terms are identical with those in Eqs. 2.4. The tension coefficient was proposed by Prof. R. V. Southwell<sup>1</sup> and was applied in the torsional analysis of space frameworks by Prof. H. Wagner.<sup>2</sup> The tension coefficients are considered as the unknowns when writing the equations of static equilibrium for a structure, instead of considering the force components as the unknowns. After the tension coefficient is determined for any member, the force components can be found from Eqs. 2.6.

The space frameworks which are commonly used in aircraft structures have bulkheads which are in parallel planes. The bulkheads resist loads in their planes but are too flexible to resist loads normal to the planes of the bulkheads. For the structure shown in Fig. 2.14(a), the bulkheads  $BCDE$  and  $B'C'D'E'$  are in parallel planes and are assumed to be loaded with equal and opposite torsional couples  $T$  in the planes of the bulkheads. From a summation of forces along the  $x$  axis at joints  $D$  and  $B'$ , the forces in members  $BB'$  and  $DD'$  are found to be zero, since the bulkheads cannot resist forces normal to their planes. The remaining members,  $EE'$ ,  $E'B$ ,  $BC'$ ,  $C'C$ ,  $CD'$ , and  $D'E$ , called the envelope members, have equal components of length  $X$ . The envelope members are also found to have equal force components  $F_x$  from a summation of forces along the  $x$  axis at joints  $E$ ,  $E'$ ,  $B$ ,  $C'$ ,  $C$ , and  $D'$ . The tension coefficients,  $\mu = F_x/X$ , are therefore equal for all the envelope members.

The value of the tension coefficient  $\mu$  for the envelope members is now obtained from the equilibrium of torsional moments about an axis normal to the bulkheads. The forces in the envelope members must be projected to the plane of the bulkhead in order to take moments about an

axis normal to the bulkheads. In Fig. 2.14(c), the lengths of the envelope members are projected to the plane of bulkhead  $BCDE$ , and the shaded area is enclosed by these projected lengths. An arbitrary point,  $O$ , is chosen as the center of moments. Figure 2.14(d) represents the

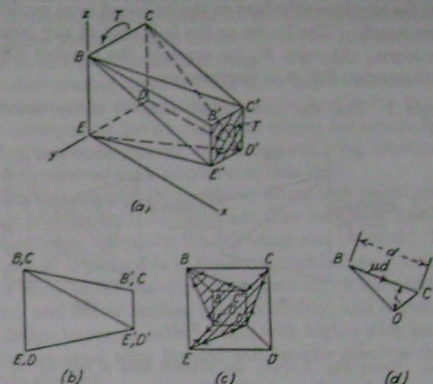


FIG. 2.14.

same projected view as Fig. 2.14(c), but the member  $BC'$  is isolated. The member  $BC'$  has a projected length  $d$  and a force component of  $\mu d$  in the plane of the bulkhead. The force component in this member has a moment  $\mu r d$ , where  $r$  is the moment arm shown. Since the area of the triangle  $OBC'$  shown in Figs. 2.14(c) or (d) is  $rd/2$ , the moment of the force in member  $BC'$  about the axis through  $O$  may be written as follows.

$$\Delta T = 2\mu \times (\text{area } OBC')$$

The moments of the forces in all the envelope members may be obtained in a similar manner. The sum of all the triangular areas will be equal to the shaded area shown in Fig. 2.14(c), and the sum of all the torsional increments will be equal to the external torque,  $T$ .

$$T = 2\mu \times (\text{area } EE'BC'CD')$$

This area, which is always equal to the area enclosed by the projection of the envelope members on the plane of a bulkhead, will be designated as  $A$ . The equation for the tension coefficient may now be written in the following form:

$$\mu = \frac{T}{2A} \quad (2.7)$$



The point  $O$  may be chosen arbitrarily, since the total area  $A$  will be the same for any position of  $O$ . The directions of the forces may be obtained by considering the forces on one end bulkhead, such as  $B'C'D'E'$ . The external load is a clockwise couple, and the forces exerted by the envelope members on the bulkhead must produce a counter-clockwise couple. The forces on the bulkhead at any point must have opposite signs. At point  $E'$ , for example, the member  $EE'$  must be in tension if member  $BE'$  is in compression.

**Example 1.** Find the forces in all members of the structure shown in Fig. 2.15.

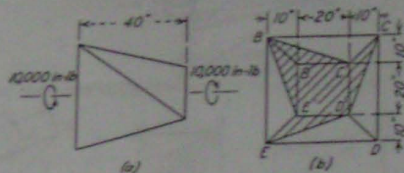


FIG. 2.15.

**Solution.** From Fig. 2.15(b), the shaded area is 800 sq in. The tension coefficient for each envelope member is

$$\mu = \frac{T}{2A} = \frac{10,000}{2 \times 800} = 6.25 \text{ lb/in.}$$

The loads are now obtained as the product of the length of the member and the tension coefficient and are shown in the following table.

The directions of the forces are determined by inspection, and are designated by positive signs for tension members and negative signs for compression members.

TABLE 2.2

| Member | Length<br>$L$ | Force<br>$\mu L$ |
|--------|---------------|------------------|
| $EE'$  | 42.5          | +266             |
| $BE'$  | 51.0          | -319             |
| $BC'$  | 51.0          | +319             |
| $CC'$  | 42.5          | -266             |
| $CD'$  | 51.0          | +319             |
| $ED'$  | 51.0          | -319             |

**Example 2.** Find the forces in members  $EE'$ ,  $BE'$ ,  $BC'$ ,  $CC'$ ,  $CD'$ , and  $ED'$  of the structure shown in Fig. 2.16. Assume rigid bulkheads in planes  $BCDE$

and  $B'C'D'E'$ . Assume also that the vertical load of 1,000 lb is equivalent to vertical loads of 500 lb on each side truss plus a torsional moment of 10,000 in-lb, as shown in Fig. 2.16.

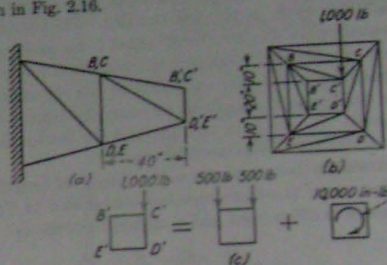


FIG. 2.16.

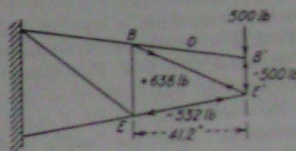


FIG. 2.17.

**Solution.** The side trusses are first analyzed as coplanar trusses carrying loads of 500 lb. The forces obtained are shown in Fig. 2.17. These forces are then combined with the forces obtained in Example 1 for the torsional loading.

TABLE 2.3

| Member | Forces resulting from<br>500 lb per side | Forces resulting from<br>10,000 in-lb, torsion | Total forces |
|--------|------------------------------------------|------------------------------------------------|--------------|
| $EE'$  | -532                                     | +266                                           | -266         |
| $BE'$  | +638                                     | -319                                           | +319         |
| $BB'$  | 0                                        | 0                                              | 0            |
| $BC'$  | 0                                        | +319                                           | +319         |
| $CC'$  | 0                                        | -266                                           | -266         |
| $CD'$  | +638                                     | +319                                           | +957         |
| $DD'$  | -532                                     | 0                                              | -532         |
| $ED'$  | 0                                        | -319                                           | -319         |

This structure is in reality statically indeterminate, and the torsional analysis gives only approximate values of the forces in the members. An exact analysis could be made by the methods used for statically



indeterminate structures, which will be treated in a later chapter. Since a statically indeterminate analysis depends on the deflections of various members, it is first necessary to know the areas of all members before making the analysis.

The approximation involved in the torsional analysis of Example 2 is the assumption that there are no resultant forces perpendicular to the plane of bulkhead  $BCDE$ . While this assumption was correct for the structure of Example 1, it is slightly in error here because the structure to the left of the bulkhead supplies additional restraints. The force components along the  $x$  axis may not be the same for all envelope members. Another distribution of forces would exist if the members in plane  $BB'EE'$  were very flexible and those in plane  $CC'DD'$  extremely rigid. The side truss in plane  $CC'DD'$  would then transfer all the 1,000-lb load into the wall at the left of the structure, with the members carrying forces twice as large as those shown in Fig. 2.17, and other members of the structure would carry no appreciable load. If the structure were supported rigidly at points  $B$ ,  $C$ ,  $D$ , and  $E$ , the torsional analysis would be considerably in error, since the loaded truss  $CC'DD'$  would carry more of the load directly to the supports.

In most aircraft structures, the rigidity is such that torsional forces may be computed separately and superimposed on the forces resulting from symmetrical loads. Rigid supports, such as those shown at the left of Fig. 2.16(a), seldom exist, and the structure would normally extend further to the left. Most bulkheads would be free to warp from their initial plane, and would have no restraining forces normal to the plane of the bulkhead. A statically indeterminate analysis, while comparatively simple for the structure with rigid supports, would be more difficult for the actual aircraft structure, and it is seldom used. Where bulkheads are not free to warp, as in the case of wing bulkheads at the center line of the airplane, local corrections are applied to the analysis made by superimposing bending and torsional forces.

**2.5. Wing Structures.** Most of the early types of airplanes were biplanes, because it was possible to design an efficient, lightweight wing structure with external bracing. The wing loading, which is the ratio of the gross weight of the airplane to the total wing area, was quite low for the early biplanes in order to permit slow landing speeds and to permit cruising at a low engine-power output. The wing weight had to be small for each square foot of wing area, because of the low wing loading. The wings were usually constructed with two wooden spanwise beams, or spars, which were braced externally to form a space truss. The spars, which supported light chordwise former ribs, were braced horizontally by occasional compression ribs with wire  $x$  bracing between them.

The structural advantages of the biplane were far outweighed by the aerodynamic disadvantages. The drag of the numerous struts and braces, and the aerodynamic interference of the wings, external bracing, and fuselage, had a more adverse effect on performance than did the additional structural weight required for monoplane construction. The early monoplane wings were also externally braced, and the construction was similar to that used for biplane wings. For light private airplanes, in which a slow landing speed is often more important than other performance features, the wing loading must be kept quite low, and the wing must consequently have a very light weight per square foot. It is possible that externally braced, fabric-covered wing construction will continue to be used for some airplanes of this type.

In military and commercial airplanes, where high speed performance and efficient cruising are more important than very slow landing speeds, only full-cantilever, internally braced wings can be used. The added structural weight is preferable to the aerodynamic drag of external brace members. With better materials and methods of construction to reduce weight, and better landing fields, visibility, and landing-gear and wing-flap design to permit safer landing with higher wing loadings, it is possible that external wing bracing will become obsolete even for the light private airplane.

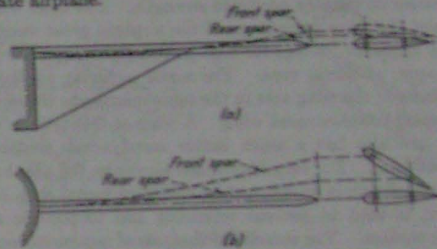


FIG. 2.18.

The air pressure on a wing changes for different flight conditions. The resultant pressure is nearer the leading edge at high angles of attack than at low angles of attack and also shifts when the ailerons or flaps are deflected. The wing must be rigid torsionally, so that it cannot twist enough to affect its aerodynamic properties when the load shifts. The simple two-spar, fabric-covered wing with a single drag truss would twist excessively if used for a full-cantilever wing. In Fig. 2.18(a), the deflections (exaggerated) of a braced wing in which the front spar deflects more than the rear spar are compared with the deflections of a full-



cantilever wing in Fig. 2.18(b) in which the front spar deflects more than the rear spar. It is seen that the angle of twist of the braced wing is negligible compared with the angle of twist of the full-cantilever wing.

In full-cantilever wings it is necessary to provide a structure which is rigid torsionally, so that both spars must deflect approximately the same distance. This can be done in a fabric-covered wing by providing drag trusses at both the upper and lower surfaces of the wing, as shown in Fig. 2.19, rather than the single drag truss shown in Figs. 2.18 and 2.20.

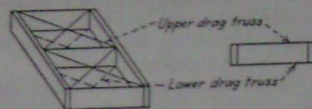


FIG. 2.19.

It is desirable to have the area of the wing cross section enclosed by the spars and the drag trusses as large as possible, since this is the area  $A$  to be used in Eq. 2.7 for a torsional analysis.

High-performance airplanes have much higher wing loadings than light private airplanes. The wing weight per square foot must be much greater in order to provide adequate strength. The total wing weight may be about the same percentage of the airplane gross weight as it is for the light airplane. Wings for high-performance airplanes are of the semimonocoque, all-metal type. The metal covering on the top and bottom surfaces of the wing acts in the same manner as the drag trusses of the two-spar, fabric-covered wing. A wing of this type is very rigid torsionally, and also has a much better aerodynamic surface than a fabric-covered wing. For very high-speed airplanes, the slight wrinkling of even the metal skin is objectionable, and thick metal skin, or plastic-coated metal, is used to prevent wrinkling and to provide a smooth aerodynamic surface. The structural analysis of semimonocoque wings is treated in detail in later chapters.

**Example 1.** Find the loads on the lift and drag-truss members of the externally braced monoplane wing shown in Fig. 2.20. The air load is assumed to be uniformly distributed along the span of the wing. The diagonal drag-truss members are wires, with the tension diagonal effective and the other diagonal carrying no load.

**Solution.** The vertical load of 20 lb/in. is distributed to the spars in inverse proportion to the distance of the center of pressure from the spars. The load on the front spar is therefore 16 lb/in., and that on the rear spar 4 lb/in. If the front spar is considered as a free body, as shown in Fig. 2.21(a), the vertical forces at  $A$  and  $G$  may be obtained.

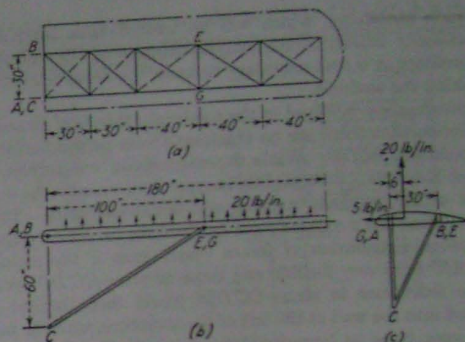


FIG. 2.20.

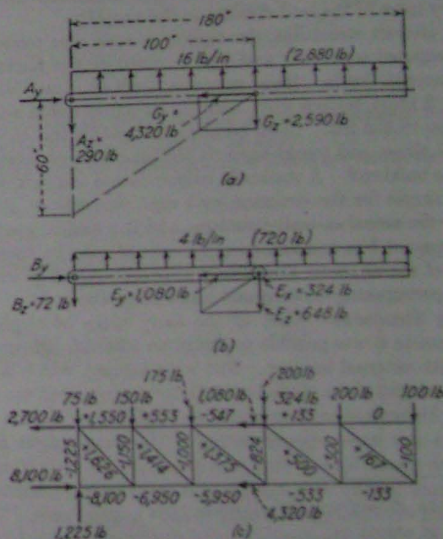


FIG. 2.21.





$$\begin{aligned}\Sigma M_A &= -11 \times 180 \times 80 + 100S_2 = 0 \\ S_2 &= 2,590 \text{ lb} \\ \frac{S_2}{180} &= \frac{2,590}{80} \\ S_2 &= 4,520 \text{ lb} \\ \Sigma F_x &= 11 \times 180 - 2,590 - A_x = 0 \\ A_x &= 290 \text{ lb}\end{aligned}$$

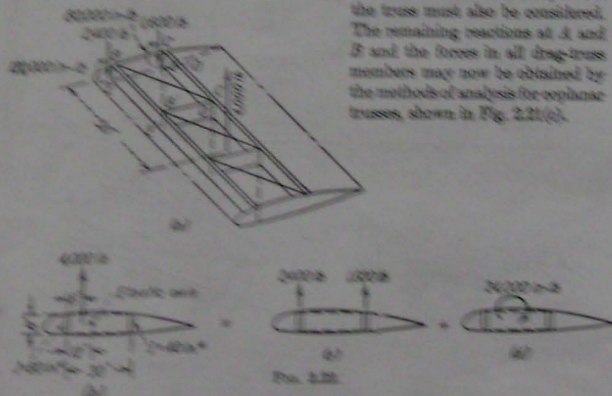
Force  $A_x$  cannot be found at this point in the analysis, since the drag-truss members exert forces on the front spar which are not shown in Fig. 2.21(a).

If the rear spar is considered as a free body, as shown in Fig. 2.21(b), the various forces at  $B$  and  $E$  may be obtained.

$$\begin{aligned}\Sigma M_B &= -4 \times 180 \times 80 + 100S_1 = 0 \\ S_1 &= 648 \text{ lb} \\ \frac{S_1}{180} &= \frac{S_2}{80} = \frac{648}{80} \\ S_2 &= 324 \text{ lb} \quad S_2 = 1,080 \text{ lb} \\ \Sigma F_x &= 4 \times 180 - 648 - B_x = 0 \\ B_x &= 72 \text{ lb}\end{aligned}$$

The loads in the plane of the drag truss may now be obtained. The forward load of 3 lb/in. is applied as concentrated loads at the panel points, as shown in Fig. 2.21(c). The components of the forces at  $B$  and  $E$  which lie in the plane of the truss must also be considered.

The remaining reactions at  $A$  and  $B$  and the forces in all drag-truss members may now be obtained by the methods of analysis for coplanar trusses, shown in Fig. 2.21(c).



**Example 2.** Find the forces acting at the cut cross section of the two-spar, half-cantilever wing shown in Fig. 2.22. The wing has two drag trusses in horizontal planes 5 in. apart, which are assumed to supply enough torsional rigidity that the two spars have equal bending deflections.

**Solution.** The bending deflections of the spars are proportional to their loads and inversely proportional to their moments of inertia. Since the rear spar has a moment of inertia of 40 in.<sup>4</sup> and the front spar a moment of inertia of 80 in.<sup>4</sup>, the rear spar must carry 40 per cent and the front spar 60 per cent of the total load in order to have the bending deflections of the beams equal. If the 4,000-lb resultant force acted at a point 40 per cent of the distance between spars from the front spar, the former ribs would distribute the loads to the spars in the correct proportion, and there would be no torsional moment producing load in the drag-truss members. The sparwise line on which a load may be applied without producing torsion is called the elastic axis, and the distances of this axis from the spars are inversely proportional to their moments of inertia. The wing is analyzed by superimposing the forces produced by torsion about the elastic axis and those produced by loads applied at the elastic axis.

The elastic axis for this wing is 12 in. from the front spar, and the torsional moment is the product of the 4,000-lb resultant air load and its moment arm about the elastic axis.

$$T = 4,000 \times 8 = 24,000 \text{ in-lb}$$

The area obtained by projecting the envelope members on the plane of the bulkhead  $BCDE$  is 240 sq in. The tension coefficient is now obtained from Eq. 2.7 as follows:

$$t = \frac{T}{A} = \frac{24,000}{2 \times 240} = 50$$

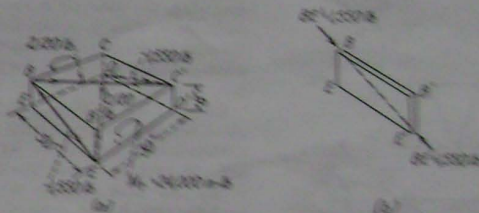


FIG. 2.22.

The forces in the envelope members of the tension structure shown in Fig. 2.22(a) are as follows:

TABLE 2.4

| Member | L, in. | Force, lb |
|--------|--------|-----------|
| BE     | 31.0   | -1,550    |
| BC     | 42.4   | +2,120    |
| DC     | 31.0   | -1,550    |
| DE     | 42.4   | +2,120    |





It will be noticed that members  $BE'$  and  $DC'$  are fictitious members which represent the portions of the spars acting as torsion structure. The loads on the actual spar members are obtained, as shown in Fig. 2.23(b), by applying the loads obtained for the fictitious members to the actual spar members.

The 4,000-lb load applied at the elastic axis is distributed by the ribs as 2,400 lb to the front spar and 1,600 lb to the rear spar. The bending moment in the front spar at the cut cross section is

$$M = 2,400 \times 50 = 120,000 \text{ in-lb}$$

The bending moment in the rear spar is

$$M = 1,600 \times 50 = 80,000 \text{ in-lb}$$

These forces are shown in Fig. 2.22(a).

**Example 3.** The wing structure shown in Fig. 2.24 is similar to that used for some full-cantilever fabric-covered wings. One main spar, member  $A_1A_2B_2B_1$ , carries all the wing bending moment. The structure aft of this spar carries wing torsion and is somewhat similar to landing-gear torque links. The structure aft of the spar also serves as a truss to resist drag loads on the wing. Unlike the two-spar wing, the structure is statically determinate, since a section through the wing would cut only six members, which could be analyzed by the equations of statics.

**Solution.** The 4,000-lb load 9 in. aft of the spar can be replaced by a 4,000-lb load in the plane of the spar and a 36,000 in-lb couple. The forces in the spar truss members resulting from the 4,000-lb load in the plane of the spar are shown in Fig. 2.25(a) and are tabulated in column (2) of Table 2.5, as the forces in the members resulting from bending of the spar.

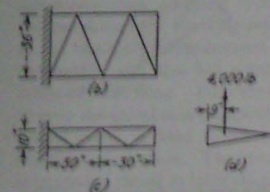


FIG. 2.24.

The envelope members of the torsion structure are shown in Fig. 2.25(b) as members  $A_1C_1$ ,  $C_1B_1$ ,  $B_1A_1$ , and  $A_2C_2$ . The tension coefficients for these members are

$$\mu = \frac{T}{2A} = \frac{36,000}{2 \times 180} = 100 \text{ lb/in.}$$

Members  $A_1C_1$  and  $B_1A_1$  have a length of 39.3 in. and forces of  $39.3 \times 100 = 3,930$  lb. Member  $A_2B_2$  has a length of 18 in. and a force of 1,800 lb. The force in member  $B_2A_2$  is 1,500 lb, and the directions of all forces are shown in Fig. 2.25(b). These forces are tabulated in column (3) of Table 2.5 as forces

TABLE 2.5

| Member<br>(1) | Forces in members, lb |                |              |
|---------------|-----------------------|----------------|--------------|
|               | Bending<br>(2)        | Torsion<br>(3) | Total<br>(4) |
| $A_1A_2$      | -18,000               | 0              | -18,000      |
| $A_1B_1$      | -7,210                | +1,800         | -5,410       |
| $B_1B_2$      | +24,000               | -1,500         | +22,500      |
| $B_2A_2$      | +7,210                | -1,800         | +5,410       |
| $B_2B_1$      | +12,000               | +1,500         | +13,500      |
| $A_2B_2$      | 0                     | 0              | 0            |
| $A_2A_1$      | -6,000                | 0              | -6,000       |
| $A_2B_1$      | -7,210                | +1,800         | -5,410       |
| $B_2B_1$      | +12,000               | -1,500         | +10,500      |
| $B_1A_2$      | +7,210                | -1,800         | +5,410       |
| $B_1B_2$      | 0                     | +1,500         | +1,500       |
| $A_1C_1$      | 0                     | -3,930         | -3,930       |
| $B_1C_1$      | 0                     | +3,930         | +3,930       |
| $A_2C_2$      | 0                     | +3,930         | +3,930       |
| $B_2C_2$      | 0                     | -3,930         | -3,930       |
| $A_1C_2$      | 0                     | -3,930         | -3,930       |
| $B_1C_2$      | 0                     | +3,930         | +3,930       |
| $A_2C_3$      | 0                     | +3,930         | +3,930       |
| $B_2C_3$      | 0                     | -3,930         | -3,930       |

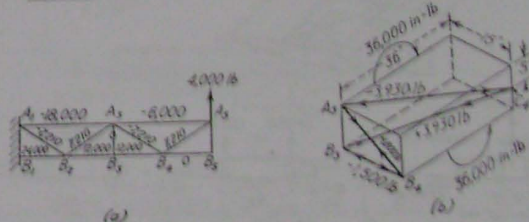


FIG. 2.25.

resulting from torsion. The resultant forces in the members from combined bending and torsion are tabulated in column (4). There are no forces in members  $C_1C_2$ ,  $C_2C_3$ , and  $C_1C_3$  for a vertical loading, but these members would resist chordwise bending from drag loads.

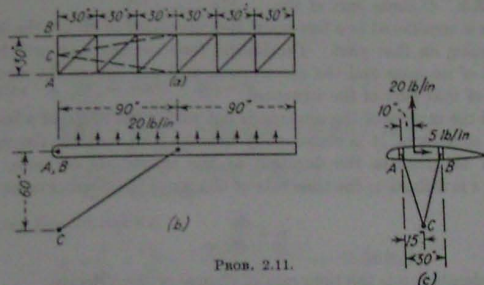
Care must be exercised in finding the load in member  $A_2B_2$ . In this case, a

summation of vertical forces at joint  $B_2$  gives no load in the member, but the member would be loaded if the torque varied along the span, or if there were drag loads acting.

## PROBLEMS

2.9. Solve Example 1, Art. 2.3, by a summation of forces along each axis, considering the tension coefficients for the three members as the unknowns. It is not necessary to find the direction cosines, as the force components will be the product of the tension coefficients and the length components. After finding the tension coefficients the forces are found as the product of the coefficients and the lengths of the members.

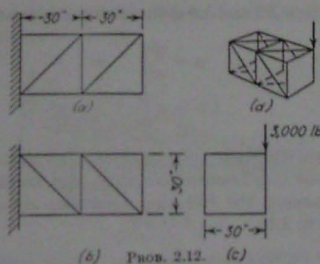
2.10. Analyze the structure of Example 1, Art. 2.5, assuming the vertical load of 20 lb/in. to act midway between the spars and the drag load of 5 lb/in. to act in the aft direction.



PROB. 2.11.

2.11. Analyze the braced-wing monoplane structure shown.

2.12. Find the loads in the fuselage truss structure shown by analyzing the side trusses as coplanar structures each carrying half the vertical load and by



PROB. 2.12.

superimposing the forces resulting from torsion about the center line. Tabulate the forces in the manner shown in Table 2.5.

2.13. Find the torsional forces in the structure of Example 3, Art. 2.5, by the method used for landing-gear torque links in Example 4, Art. 2.3. After finding the forces on the aft structure, apply these forces to the spar, and analyze the spar as a coplanar truss. The results should check the values in column (3), Table 2.5.

## REFERENCES FOR CHAPTER 2

1. SOUTHWELL, R. V.: Primary Stress Determination in Space Frames, *Engineering*, Feb. 6, 1920.
2. WAGNER, H.: The Analysis of Aircraft Structures as Space Frameworks, *NACA TM 522*, 1929.
3. NILES, A. S., and J. S. NEWELL: "Airplane Structures," Vol. I, Chap. 8, John Wiley & Sons, Inc., New York, 1943.



## CHAPTER 3

## INERTIA FORCES AND LOAD FACTORS

**3.1. Pure Translation.** The maximum load on any part of the airplane structure occurs when the airplane is being accelerated. The loads produced by landing impact or when maneuvering or encountering gusts in flight are always greater than the loads occurring when all the forces on the airplane are in equilibrium. Before any member can be designed, it is therefore necessary to determine the inertia forces acting on the structure. If the inertia forces are included, it is possible to draw a free-body diagram for any member showing the forces in equilibrium.

In many of the loading conditions, the airplane may be considered as being in pure translation, since the rotational velocities and accelerations

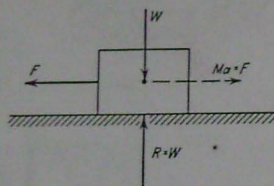


FIG. 3.1.

are small. The inertia force on any element of mass is equal to the product of the mass and the acceleration and acts in a direction opposite to the acceleration. If the applied loads and inertia forces act on an element as a free body, these forces are in equilibrium. If a force  $F$  acts on a block on a frictionless plane, as shown in Fig. 3.1, the block will be

$$F = Ma \quad (3.1)$$

In engineering problems, the common unit of mass is the slug. The mass in slugs is obtained by dividing the weight in pounds by the acceleration of gravity  $g$  in feet per second per second.

$$M = \frac{W \text{ lb-sec}^2}{g \text{ ft}} \quad (3.2)$$

If  $g$  is 32.2 ft/sec<sup>2</sup>, the value normally used, a body having a mass of 1 slug will have a weight of 32.2 lb. In many problems the inertia force  $Ma$  can be found as a force in pounds by the equations of static equilibrium, without first finding the mass and the acceleration.

The airplane shown in Fig. 3.2 is moving forward after landing, and has a braking force  $F$  applied to the right. The acceleration is to the right, and the inertia force on each element of mass is to the left, or opposite to the direction of acceleration. The sum of the inertia forces on all elements will be the product of the total mass of the airplane and the acceleration, or  $Ma$ .

Since the inertia forces are distributed in proportion to the mass  $dM$  of each element, the resultant force will act at the center of gravity of the airplane, as shown in Fig. 3.2. If some part of the

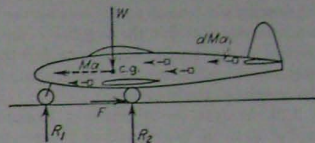


FIG. 3.2.

airplane is considered as a free body, it is necessary to obtain the inertia force acting on that part. The inertia force on any part will be the product of its mass and the acceleration and will act at the center of gravity of that part of the structure.

While the motion of the airplane is not within the scope of a book on aircraft structures, it is occasionally necessary to consider the motion in order to estimate the duration or the magnitude of loads. The velocity  $v$  is defined as the time rate of change of the displacement  $s$ .

$$v = \frac{ds}{dt} \quad (3.3)$$

The acceleration  $a$  is the time rate of change of the velocity,

$$a = \frac{dv}{dt} \quad (3.4)$$

A combination of Eqs. 3.3 and 3.4 gives other forms for the acceleration:

$$a = \frac{d^2s}{dt^2} \quad (3.5)$$

$$a = v \frac{dv}{ds} \quad (3.6)$$

For a motion of pure translation of a rigid body, all elements of the body must have the same displacement, velocity, and acceleration. If the acceleration is constant, the following equations are obtained by integrating Eqs. 3.4 to 3.6:

$$v - v_0 = at \quad (3.7)$$

$$s = v_0 t + \frac{1}{2} at^2 \quad (3.8)$$

$$v^2 - v_0^2 = 2as \quad (3.9)$$



where  $s$  is the distance moved in time  $t$ ,  $v_0$  is the initial velocity, and  $v$  is the final velocity after  $t$  sec.

**Example 1.** Assume that the airplane shown in Fig. 3.3 weighs 20,000 lb. and that the braking force  $F$  is 8,000 lb.

a. Find the wheel reactions  $R_1$  and  $R_2$ .

b. Find the landing run if the airplane lands at 100 mph (146.7 ft/sec).

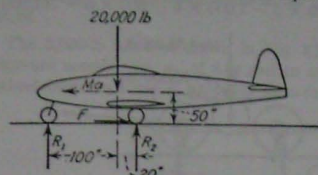


FIG. 3.3.

**Solution.** a.  $\Sigma F_x = 8,000 - Ma = 0$

$$Ma = 8,000 \text{ lb}$$

$$\Sigma M_{R_2} = 120R_1 - 8,000 \times 50 - 20,000 \times 20 = 0$$

$$R_1 = 6,670 \text{ lb}$$

$$\Sigma F_y = 6,670 - 20,000 + R_2 = 0$$

$$R_2 = 13,330 \text{ lb}$$

b. From Eqs. 3.1 and 3.2,

$$a = \frac{F}{M} = \frac{Fg}{W} = \frac{-8,000 \times 32.2}{20,000} = -12.88 \text{ ft/sec}^2$$

From Eq. 3.9,

$$\begin{aligned} v^2 - v_0^2 &= 2as \\ 0 - (146.7)^2 &= 2(-12.88)s \\ s &= 835 \text{ ft} \end{aligned}$$

When  $s$  is measured as positive to the left, it is necessary to consider  $a$  as negative, since it is an acceleration to the right.

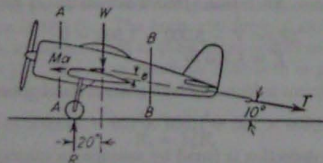


FIG. 3.4.

**Example 2.** When landing on a carrier, a 10,000-lb airplane is given a deceleration of  $3g$  (96.6 ft/sec<sup>2</sup>) by means of a cable engaged by an arresting hook, as shown in Fig. 3.4.

a. Find the tension in the cable, the wheel reaction  $R$ , and the distance  $e$  from the center of gravity to the line of action of the cable.

b. Find the tension in the fuselage at vertical sections AA and BB if the portion of the airplane forward of section AA weighs 3,000 lb and the portion aft of section BB weighs 1,000 lb.

c. Find the landing run if the landing speed is 80 ft/sec.

**Solution.** a. First considering the entire airplane as a free body,

$$Ma = \frac{W}{g} a = \frac{10,000}{g} \times 3g = 30,000 \text{ lb}$$

$$\Sigma F_x = T \cos 10^\circ - 30,000 = 0$$

$$T = 30,500 \text{ lb}$$

$$\Sigma F_y = R - 10,000 - 30,500 \sin 10^\circ = 0$$

$$R = 15,300 \text{ lb}$$

$$\Sigma M_{cg} = 20 \times 15,300 - 30,500e = 0$$

$$e = 10 \text{ in.}$$

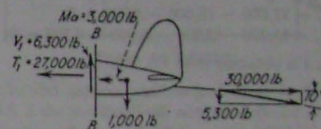


FIG. 3.5.

b. Considering the aft section of the fuselage as a free body as shown in Fig. 3.5, it is acted upon by an inertia force of

$$Ma = \frac{1,000}{g} \times 3g = 3,000 \text{ lb}$$

The tension on section BB is found as follows:

$$\Sigma F_x = 30,000 - 3,000 - T_1 = 0$$

$$T_1 = 27,000 \text{ lb}$$

Since there is no vertical acceleration, there is no vertical inertia force. Section BB has a shear force  $V_1$  of 6,300 lb, which is equal to the sum of the weight and the vertical component of the cable force.

Considering the portion of the airplane forward of section AA as a free body, as shown in Fig. 3.6, the inertia force is

$$Ma = \frac{3,000}{g} \times 3g = 9,000 \text{ lb}$$

$$\Sigma F_x = T_2 - 9,000 = 0$$

$$T_2 = 9,000 \text{ lb}$$

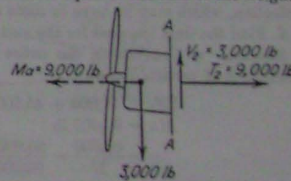


FIG. 3.6.

The section AA must also resist a shearing force  $V_2$  of 3,000 lb and a bending moment obtained by taking moments of the forces shown in Fig. 3.6.



The forces  $T_1$ ,  $T_2$ ,  $V_1$ , and  $V_2$  may be checked by considering the equilibrium of the center portion of the airplane, as shown in Fig. 3.7.

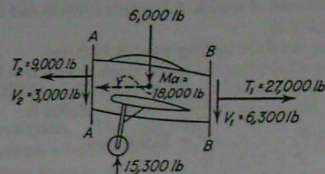


FIG. 3.7.

$$Ma = \frac{6,000}{g} \times 3g = 18,000 \text{ lb}$$

$$\Sigma F_x = 27,000 - 18,000 - 9,000 = 0$$

$$\Sigma F_y = 15,300 - 3,000 - 6,000 - 6,300 = 0$$

c. The landing run  $s$  is obtained from Eq. 3.9.

$$\begin{aligned} v^2 - v_0^2 &= 2as \\ 0 - (80)^2 &= 2(-96.6)s \\ s &= 33 \text{ ft} \end{aligned}$$

**Example 3.** A 30,000-lb airplane is shown in Fig. 3.8(a) at the time of landing impact, when the ground reaction on each main wheel is 45,000 lb.

a. If one wheel and tire weighs 500 lb, find the compression  $C$  and bending moment  $m$  in the oleo strut, if the strut is vertical and is 6 in. from the center line of the wheel, as shown in Fig. 3.8(b).

b. Find the shear and bending moment at section AA of the wing, if the wing outboard of this section weighs 1,500 lb and has its center of gravity 120 in. outboard of section AA.

c. Find the required shock strut deflection if the airplane strikes the ground with a vertical velocity of 12 ft/sec and has a constant vertical deceleration until the vertical velocity is zero. This neglects the energy absorbed by the tire deflection, which may be large in some cases.

d. Find the time required for the vertical velocity to become zero.

**Solution.** a. Considering the entire airplane as a free body and taking a summation of vertical forces,

$$\Sigma F_y = 45,000 + 45,000 - 30,000 - Ma = 0$$

$$Ma = 60,000 \text{ lb}$$

$$a = \frac{60,000}{M} = \frac{60,000g}{30,000} = 2g$$

Considering the landing gear as a free body, as shown in Fig. 3.8(b), the inertia force is

$$M_1 a = \frac{w_1}{g} a = \frac{500}{g} \times 2g = 1,000 \text{ lb}$$

The compression load  $C$  in the oleo strut is found from a summation of vertical forces.

$$\begin{aligned} \Sigma F_y &= 45,000 - 500 - 1,000 - C = 0 \\ C &= 43,500 \text{ lb} \end{aligned}$$

The bending moment  $m$  is found as follows:

$$m = 45,000 \times 6 - 1,000 \times 6 - 500 \times 6 = 261,000 \text{ in-lb}$$

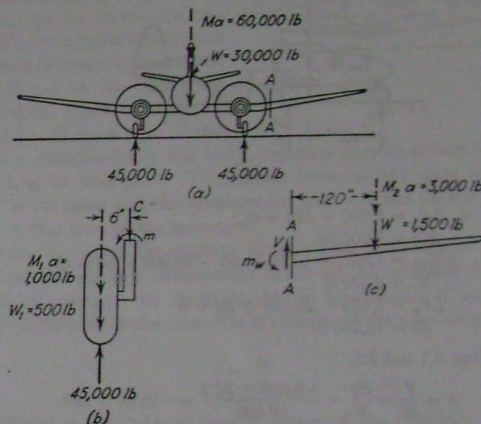


FIG. 3.8.

b. The inertia force acting on the portion of the wing shown in Fig. 3.8(c) is

$$M_2 a = \frac{w_2}{g} a = \frac{1,500}{g} \times 2g = 3,000 \text{ lb}$$

The wing shear at section AA is found from a summation of vertical forces.

$$\begin{aligned} \Sigma F_y &= V - 3,000 - 1,500 = 0 \\ V &= 4,500 \text{ lb} \end{aligned}$$

The wing bending moment is found by taking moments about section AA.

$$m_w = 3,000 \times 120 + 1,500 \times 120 = 540,000 \text{ in-lb}$$

c. The shock strut deflection is found by assuming a constant vertical acceleration of  $-2g$ , or  $-64.4 \text{ ft/sec}^2$ , from an initial vertical velocity of 12 ft/sec to a final zero vertical velocity.

$$\begin{aligned} v^2 - v_0^2 &= 2as \\ 0 - (12)^2 &= 2(-64.4)s \\ s &= 1.12 \text{ ft} \end{aligned}$$



- d. The time required to absorb the landing shock is found from Eq. 3.7.

$$\begin{aligned} v - v_0 &= at \\ 0 - 12 &= -64.4t \\ t &= 0.186 \text{ sec} \end{aligned}$$

Since this landing shock occurs for such a short interval of time, it may be less injurious to the structure and less disagreeable to the passengers than a sustained load would be.

**Example 4.** The 8,000-lb airplane shown in Fig. 3.9 is landing on soft ground with an upward acceleration,  $a_y$ , of  $3.5g$  and an aft acceleration,  $a_x$ , of  $1.5g$ . Find the wheel reactions  $A$  and  $B$ , assuming them to be parallel.

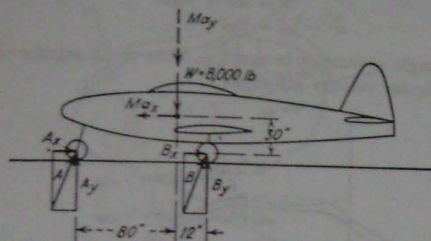


FIG. 3.9.

**Solution.** The vertical inertia force is found from the vertical acceleration.

$$Ma_y = \frac{8,000}{g} 3.5g = 28,000 \text{ lb}$$

This force acts in a direction opposite to the acceleration, or downward. The horizontal inertia force is found in a similar manner, and acts forward at the center of gravity.

$$Ma_x = \frac{8,000}{g} \times 1.5g = 12,000 \text{ lb}$$

The total vertical force at the center of gravity is the sum of the inertia force and the weight, or 36,000 lb. The horizontal force of 12,000 lb is one-third of this vertical force, so the resultant of these forces acts at an angle with the vertical which is  $\tan^{-1} \frac{1}{3}$ . The wheel reactions must both be parallel to this resultant, and the following relations are obtained:

$$\begin{aligned} A_x &= \frac{1}{3} A_y \\ B_x &= \frac{1}{3} B_y \end{aligned}$$

The forces may now be found from the equations of statics:

$$\begin{aligned} \Sigma M_B &= 92A_y - 12,000 \times 30 - 36,000 \times 12 = 0 \\ A_y &= 8,600 \text{ lb} \\ \Sigma F_x &= 8,600 - 36,000 + B_x = 0 \\ B_x &= 27,400 \end{aligned}$$

$$A_x = \frac{1}{3} A_y = 2,870 \text{ lb}$$

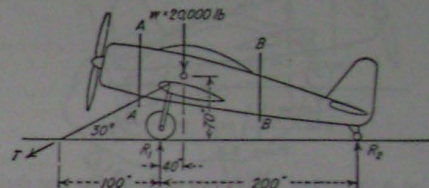
$$B_x = \frac{1}{3} B_y = 9,130 \text{ lb}$$

$$\text{Check: } \Sigma M_A = 36,000 \times 80 - 12,000 \times 30 - 27,400 \times 92 = 0$$

### PROBLEMS

3.1. A 20,000-lb airplane with the dimensions shown is being catapulted from a carrier with a forward acceleration of  $3g$ .

- a. Find the tension in the launching cable  $T$  and the wheel reactions  $R_1$  and  $R_2$ .



PROB. 3.1 AND 3.2.

b. Find the horizontal tension and vertical shear in the fuselage at cross sections  $AA$  and  $BB$  if the portion of the airplane forward of section  $AA$  weighs 3,500 lb and the portion aft of section  $BB$  weighs 2,000 lb. Check the results by considering the part of the airplane between the two cross sections.

c. If the flying speed is 80 mph, what distance is required for the airplane to gain this speed? How much time is required?

3.2. If an airplane with the dimensions used in Prob. 3.1 is taxiing, what maximum braking force may be applied at the main wheels without the airplane nosing over?

3.3. An airplane weighing 5,000 lb strikes an upward gust of air which produces a wing lift of 25,000 lb. What tail load  $P$  is required to prevent a pitching accel-

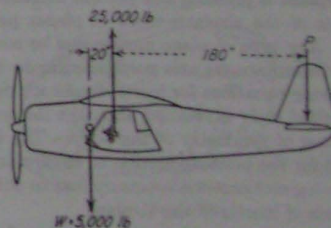


FIG. 3.3.

eration, if the dimensions are as shown? What will be the vertical acceleration of the airplane? If this lift force acts until the airplane obtains a vertical velocity of 20 ft/sec, how much time is required?



3.4. An airplane weighing 8,000 lb has an upward acceleration of  $3g$  when landing. If the dimensions are as shown, what are the wheel reactions  $R_1$  and  $R_2$ ? What time is required to decelerate the airplane from a vertical velocity

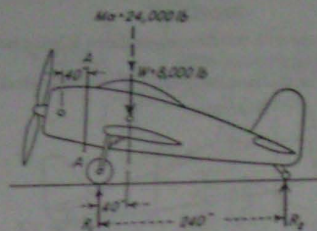


FIG. 3.4.

of 12 ft/sec? What is the vertical compression of the landing gear during this deceleration? What is the shear and bending moment on a vertical section  $AA$ , if the weight forward of this section is 2,000 lb and has a center of gravity 40 in. from this cross section?

**3.2. Inertia Forces on Rotating Bodies.** In the preceding discussion all parts of the rigid body were moving in straight, parallel lines and had equal velocities and accelerations. In many engineering problems it is necessary to consider the inertia forces acting on a rigid body which has other types of motion. In many cases where the elements of a rigid body are moving in curved paths they are moving in such a way that each element moves in only one plane, and all elements move in parallel planes. This type of motion is called plane motion, and occurs, for example, when an airplane is pitching and yet has no rolling or yawing motion. All elements of the airplane move in planes parallel to the plane of symmetry. Any type of plane motion can be considered as a rotation about some instantaneous axis perpendicular to the planes of motion, and the following equations for inertia forces are derived on the assumption that the rigid body is rotating about an instantaneous axis perpendicular to a plane of symmetry of the body. The inertia forces obtained may be used for the pitching motion of an airplane, but when used for rolling or yawing motions it is necessary first to obtain the principal axes and moments of inertia of the airplane.

A rigid body in plane motion must have the same angular velocity at all points. This angular velocity  $\omega$  is defined as the time rate of change of the angle  $\theta$  measured between a reference line on the body and a fixed reference axis.

$$\omega = \frac{d\theta}{dt} \quad \text{rad/sec} \quad (3.10)$$

The angle is expressed in radians, and  $\omega$  is in units of radians per second. The angular acceleration  $\alpha$  is the time rate of change of the angular velocity.

$$\alpha = \frac{d\omega}{dt} \quad \text{rad/sec}^2 \quad (3.11)$$

All points of a rigid body will have the same angular acceleration. For constant angular acceleration  $\alpha$ , the following expressions similar to those in Eqs. 3.7 to 3.9 are obtained:

$$\omega = \omega_0 + \alpha t \quad (3.12)$$

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2 \quad (3.13)$$

$$\omega^2 - \omega_0^2 = 2\alpha\theta \quad (3.14)$$

where  $\theta$  is the angle of rotation in time  $t$ ,  $\omega_0$  is the initial angular velocity, and  $\omega$  is the angular velocity after  $t$  sec.

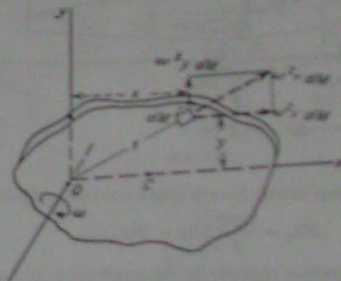


FIG. 3.10.

The rigid mass shown in Fig. 3.10 is rotating about point  $O$  with a constant angular velocity  $\omega$ . The acceleration of any point a distance  $r$  from the center of rotation is  $\omega^2 r$ , and is directed toward the center of rotation. The inertia force acting on an element of mass  $dM$  is the product of the mass and the acceleration, or  $\omega^2 r dM$ , and is directed away from the axis of rotation. This inertia force has components  $\omega^2 x dM$  parallel to the  $x$  axis and  $\omega^2 y dM$  parallel to the  $y$  axis. If the  $x$  axis is chosen through the center of gravity  $C$ , the forces are simplified. The resultant inertia force in the  $y$  direction for the entire body is found as follows:

$$F_y = \int \omega^2 y dM = \omega^2 \int y dM = 0$$

The angular velocity  $\omega$  is constant for all elements of the body, and the integral is zero because the  $x$  axis was chosen through the center of





of slug-ft<sup>2</sup>, or since the slug has units of lb-sec<sup>2</sup>/ft, the moment of inertia is in units of lb-sec<sup>2</sup>-ft.

$$I = \int r^2 dM = \int \frac{r^2}{g} dW = \frac{\text{ft}^2 \cdot \text{lb}}{\text{ft/sec}^2} = \text{lb-sec}^2 \cdot \text{ft}$$

The value of the acceleration of gravity  $g$  is used as 32.2 ft/sec<sup>2</sup>. The angular acceleration  $\alpha$  has units of radians per second per second, or since radians are dimensionless, has the dimension sec<sup>-2</sup>. The couple  $I\alpha$  is in units of foot-pounds.

All drawings of airplanes are dimensioned in units of inches, regardless of the size of the airplane. Most calculations for weight and balance, and for structures, are made using units of inches. The acceleration of gravity  $g$  is  $32.2 \times 12 = 386 \text{ in./sec}^2$ . It is convenient to calculate the moment of inertia of an airplane by multiplying the weight of each element by the square of the distance in inches and dividing by  $g = 386 \text{ in./sec}^2$ .

$$I = \frac{r^2 dW}{g} = \frac{\text{in}^2 \cdot \text{lb}}{\text{in./sec}^2} = \text{lb-sec}^2 \cdot \text{in.}$$

The inertia couple  $I\alpha$  will then be in units of inch-pounds.

**Example 1.** A 60,000-lb airplane with a tricycle landing gear makes a hard two-wheel landing in soft ground, so that the vertical ground reaction is 270,000 lb and the horizontal ground reaction is 90,000 lb. The moment of inertia about the center of gravity is 5,000,000 lb-sec<sup>2</sup>-in., and the dimensions are shown in Fig. 3.14.

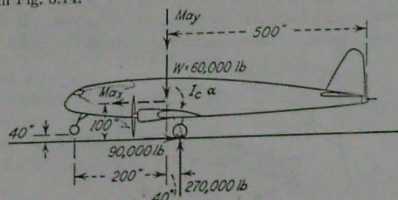


FIG. 3.14.

- Find the inertia forces on the airplane.
- Find the inertia forces on a 400-lb gun turret in the tail, which is 500 in. aft of the center of gravity. Neglect the moment of inertia of the turret about its own center of gravity.
- If the nose wheel is 40 in. from the ground when the main wheels touch the ground, find the angular velocity of the airplane and the vertical velocity of the nose wheel when the nose wheel reaches the ground, assuming no appreciable change in the moment arms. The airplane center of gravity has a vertical

velocity of 12 ft/sec at the moment of impact, and the ground reactions are assumed constant until the vertical velocity reaches zero, at which time the vertical ground reaction becomes 60,000 lb and the horizontal ground reaction becomes 20,000 lb.

**Solution.** a. The inertia forces on the entire airplane may be considered as horizontal and vertical forces  $Ma_x$  and  $Ma_y$  at the center of gravity and a couple  $I\alpha$ , as shown in Fig. 3.14. These correspond to the inertia forces shown on the mass of Fig. 3.12, since the forces at the center of gravity represent the product of the mass and the acceleration components of the center of gravity.

$$\Sigma F_x = 90,000 - Ma_x = 0$$

$$Ma_x = 90,000 \text{ lb}$$

$$\Sigma F_y = 270,000 - 60,000 - Ma_y = 0$$

$$Ma_y = 210,000 \text{ lb}$$

$$\Sigma M_G = -270,000 \times 40 - 90,000 \times 100 + I\alpha = 0$$

$$I\alpha = 19,800,000 \text{ in-lb}$$

$$a_x = \frac{90,000}{M} = \frac{90,000}{60,000} g = 1.5g$$

$$a_y = \frac{210,000}{M} = \frac{210,000}{60,000} g = 3.5g$$

$$\alpha = \frac{I\alpha}{I} = \frac{19,800,000}{5,000,000} = 3.96 \text{ rad/sec}^2$$

b. The acceleration of the center of gravity of the airplane is now known, and the acceleration and inertia forces for the turret can be obtained by the method shown in Fig. 3.13, where the center of gravity of the airplane corresponds to

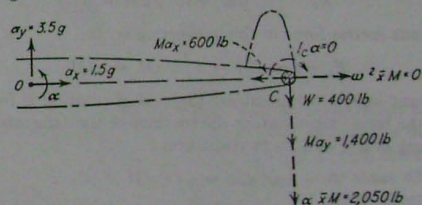


FIG. 3.15.

point  $O$  of Fig. 3.13 and the center of gravity of the turret corresponds to point  $C$ . These forces are shown in Fig. 3.15, and have the following values:

$$Ma_x = \frac{400}{g} \times 1.5g = 600 \text{ lb}$$

$$Ma_y = \frac{400}{g} \times 3.5g = 1,400 \text{ lb}$$

$$\alpha \bar{I} M = 3.96 \times 500 \times \frac{400}{386} = 2,050 \text{ lb}$$

In calculating the term  $\omega^2 M$ ,  $z$  is in inches, and  $g$  is used as 386 in./sec<sup>2</sup>. If  $z$  is in feet,  $g$  will be 32.2 ft/sec<sup>2</sup>.

The total force on the turret is 800 lb forward and 3,850 lb down. This total force is seen to be almost ten times the weight of the turret.

$c$ . The center of gravity of the airplane is decelerated vertically at 3.5*g*, or 112.7 ft/sec<sup>2</sup>. The time of deceleration from an initial velocity of 12 ft/sec to a zero vertical velocity is found from Eq. 3.7.

$$\begin{aligned} z - z_0 &= at \\ 0 - 12 &= -112.7t \\ t &= 0.106 \text{ sec} \end{aligned}$$

During this time the center of gravity moves through a distance found from Eq. 3.5.

$$\begin{aligned} z &= at + \frac{1}{2}at^2 \\ &= 12 \times 0.106 - \frac{1}{2} \times 112.7 \times (0.106)^2 \\ &= 0.630 \text{ ft, or } 7.56 \text{ in.} \end{aligned}$$

The angular velocity of the airplane at the end of 0.106 sec after the landing is found from Eq. 3.12.

$$\begin{aligned} \omega - \omega_0 &= at \\ \omega - 0 &= 3.96 \times 0.106 \\ \omega &= 0.42 \text{ rad/sec} \end{aligned}$$

The angle of rotation during this time is found from Eq. 3.13.

$$\begin{aligned} \theta_1 &= \omega_0 t + \frac{1}{2}at^2 \\ \theta_1 &= 0 + \frac{1}{2}(3.96)(0.106)^2 = 0.0222 \text{ rad} \end{aligned}$$

The vertical motion of the nose wheel resulting from this rotation, shown in Fig. 3.15, is

$$z_1 = \theta_1 r = 0.0222 \times 200 = 4.44 \text{ in.}$$

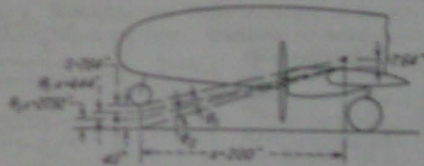


FIG. 3.15.

The distance of the nose wheel from the ground, after the vertical velocity of the center of gravity of the airplane has become zero, is

$$z_2 = 10 - 7.56 - 4.44 = 27.92 \text{ in.}$$

The remaining angle of rotation  $\theta_2$  shown in Fig. 3.16, is

$$\theta_2 = \frac{\theta_1}{z} = \frac{0.0222}{27.92} = 0.1396 \text{ rad}$$

Since the ground reactions decrease by the ratio of  $\frac{60,000}{270,000}$  after the vertical

acceleration of the airplane becomes zero, the angular acceleration decreases in the same proportion, as found by equating moments about the center of gravity.

$$\alpha_2 = \frac{60,000}{270,000} \times 3.96 = 0.88 \text{ rad/sec}^2$$

The angular velocity of the airplane at the time the nose wheel strikes the ground is found from Eq. 3.14.

$$\begin{aligned} \omega^2 - \omega_0^2 &= 2\alpha\theta \\ \omega^2 - (0.42)^2 &= 2 \times 0.88 \times 0.1396 \\ \omega &= 0.65 \text{ rad/sec} \end{aligned}$$

Since at this time the motion is rotation, with no vertical motion of the center of gravity, the vertical velocity of the nose wheel is found as follows:

$$\begin{aligned} v &= \omega z \\ v &= 0.65 \times \frac{5}{8} = 10.5 \text{ ft/sec} \end{aligned}$$

This velocity is smaller than the initial sinking velocity of the airplane. The nose wheel consequently would strike the ground with a higher velocity in a three-wheel level landing.

It is of interest to find the centrifugal force on the turret,  $\omega^2 M$ , at the time the nose wheel strikes the ground. This force was zero at the time the main wheels hit, because the angular velocity  $\omega$  was zero. For the final value of  $\omega$ , the following value is obtained

$$\omega^2 M = (0.65)^2 \times 800 \times \frac{11}{8} = 219 \text{ lb}$$

This force is much smaller than other forces acting on the turret, and is usually neglected.

In part (c) of this problem, certain simplifying assumptions were made which do not quite correspond with actual landing conditions. Aerodynamic forces were neglected, and the ground reactions on the landing gear were assumed constant while the landing gear was being compressed. The actual compression of the landing gear is a combination of the tire deflection, in which the load is approximately proportional to the deflection, and the strut deflection, in which the load is almost constant during the entire deflection, as assumed in the problem. The tire deflection may be as much as one-third to one-half the total deflection. The aerodynamic forces, which have been neglected, would probably reduce the maximum angular velocity of the airplane, since the horizontal tail moves upward as the airplane pitches, and the combination of upward and forward motion would give a downward aerodynamic force on the tail, tending to reduce the pitching acceleration.

The aerodynamic effects of the lift on the wing and tail surfaces are not shown in Fig. 3.14, but they will not affect the pitching acceleration appreciably if the ground reactions remain the same. Just before the airplane strikes the ground, the lift forces on the wing and tail are in equilibrium with the gravity force of 60,000 lb. Since the horizontal velocity of the airplane and the angle of attack are not appreciably changed ( $\theta = 0.0222 \text{ rad} = 1.27^\circ$ ), the lift forces continue



to balance the weight of the airplane when the center of gravity is being decelerated. Instead of the weight of 60,000 lb shown in Fig. 3.14, there should be an additional inertia force of 60,000 lb down at the center of gravity. The moments about the center of gravity and the pitching acceleration are not changed, but the vertical deceleration  $a_y$  is increased. At the end of the deceleration of the center of gravity the ground reactions are almost zero, since most of the airplane

weight is carried by the lift on the wings. The airplane then pitches forward through the angle  $\theta_2$ , which appreciably changes the angle of attack ( $\theta_2 = 0.1396$  rad =  $8^\circ$ ). The wing lift is then decreased, and most of the weight is supported by the ground reactions on the wheels. For the structural design of the airplane, only the loads during the initial impact are usually significant.

**Example 2.** An airplane weighing 8,000 lb is flying at 250 mph in a

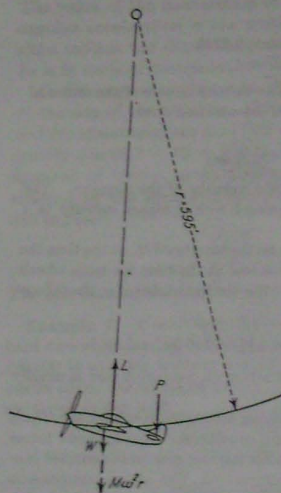


FIG. 3.17.

vertical circle with a radius of 595 ft, as shown in Fig. 3.17. Find the lift  $L$  on the wing and the load  $P$  on the tail at the time the airplane is moving horizontally, if the dimensions are as shown in Fig. 3.18.

**Solution.** If the airplane is moving at a constant velocity along a circular path, the angular acceleration is zero, and the only inertia force is the centrifugal force which is found as follows:

$$250 \text{ mph} = 366 \text{ ft/sec}$$

$$\omega = \frac{v}{r} = \frac{366}{595} = 0.615 \text{ rad/sec}$$

$$Ma_y = \frac{8,000}{32.2} \times (0.615)^2 \times 595 = 56,000 \text{ lb}$$

This inertia force acts down at the center of gravity of the airplane and must be added to the weight of the airplane. The propeller thrust is assumed to be equal

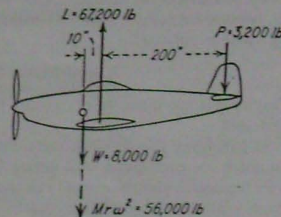


FIG. 3.18.

and opposite to the airplane drag, so that only the vertical forces need be considered. The forces  $L$  and  $P$  are obtained from the equations of statics.

$$\Sigma M_L = 200P - 8,000 \times 10 - 56,000 \times 10 = 0$$

$$P = 3,200 \text{ lb}$$

$$\Sigma F_y = L - 56,000 - 8,000 - 3,200 = 0$$

$$L = 67,200 \text{ lb}$$

**Example 3.** The airplane in Example 2 is maneuvered by giving the control stick an abrupt forward displacement so that the airplane is given a pitching acceleration of 6 rad/sec<sup>2</sup>. Find the tail load and the inertia forces if the wing lift does not change while the control stick is being moved. The moment of inertia of the airplane about a pitching axis through the center of gravity is 180,000 lb-sec<sup>2</sup>-in. Find the forces acting on the engine which weighs 1,000 lb and is 50 in. forward of the center of gravity. Find the time required for the airplane to pitch through an angle of  $3^\circ$ , if the pitching acceleration is constant.

**Solution.** The inertia couple is found as follows:

$$I\alpha = 180,000 \times 6 = 1,080,000 \text{ in-lb}$$

The tail load  $P_1$  and the inertia force at the center of gravity are shown in Fig. 3.19 and are found as follows:

$$\Sigma M_{cg} = 1,080,000 - 67,200 \times 10 - 210P_1 = 0$$

$$P_1 = 1,940 \text{ lb}$$

$$\Sigma F_y = 67,200 + 1,940 - 8,000 - Ma_y = 0$$

$$Ma_y = 61,140 \text{ lb}$$

$$a_y = \frac{61,140}{M} = \frac{61,140}{8,000} g = 7.65g$$

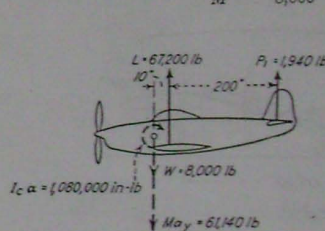


FIG. 3.19.

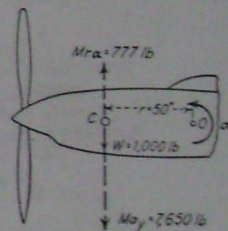


FIG. 3.20.

The forces acting on the engine, shown in Fig. 3.20, consist of the weight of 1,000 lb down, the inertia force  $Ma_y$  down, and the inertia force  $Mr\alpha$  up.

$$Ma_y = \frac{1,000}{g} \times 7.65g = 7,650 \text{ lb}$$

$$Mr\alpha = \frac{1,000}{386} \times 50 \times 6 = 777 \text{ lb}$$





sider the combined weight and inertia forces. When the airplane is being accelerated upward, the weight and inertia forces add directly. The weight  $w$  of any part and the inertia force  $wa/g$  have a sum  $nw$ .

$$nw = w + w \frac{a}{g}$$

or

$$n = 1 + \frac{a}{g} \quad (3.20)$$

The combined inertia and gravity forces are considered in the analysis in the same manner as weights which are multiplied by the load factor  $n$ .

In the case of an airplane in flight with no horizontal acceleration, as shown in Fig. 3.23, the propeller thrust is equal to the airplane drag, and

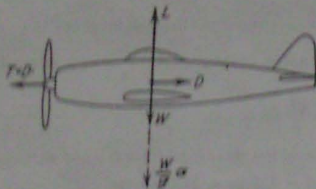


FIG. 3.23.

the horizontal components of the inertia and gravity forces are zero. The weight and the inertia force on the airplane act down and will be equal to the lift. The airplane lift  $L$  is the resultant of the wing and tail lift forces. The load factor is defined as follows:

$$\text{Load factor} = \frac{\text{lift}}{\text{weight}}$$

or

$$n = \frac{L}{W} \quad (3.21)$$

This value for the load factor can be shown to be the same as that given by Eq. 3.20 by equating the lift  $nW$  to the sum of the weight and inertia forces:

$$L = nW = W + W \frac{a}{g}$$

or

$$n = 1 + \frac{a}{g}$$

which corresponds to Eq. 3.20.

An airplane frequently has horizontal acceleration as well as vertical acceleration. The airplane shown in Fig. 3.24 is being accelerated forward, since the propeller thrust  $T$  is greater than the airplane drag  $D$ . Every element of mass in the airplane is thus under the action of a

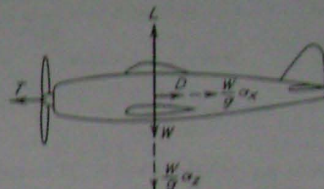


FIG. 3.24.

horizontal inertia force equal to the product of its mass and the horizontal acceleration. It is also convenient to consider the horizontal inertia loads as equal to the product of a load factor  $n_x$  and the weights. This horizontal load factor, often called the thrust load factor, is obtained from the equilibrium of the horizontal forces shown in Fig. 3.24.

$$n_x W = \frac{\alpha_x}{g} W = T - D$$

or

$$n_x = \frac{T - D}{W} \quad (3.22)$$

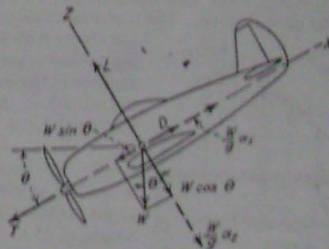


FIG. 3.25.

A more general case of translational acceleration is that shown in Fig. 3.25, in which the airplane thrust line is not horizontal. It is usually convenient to obtain components of forces along  $x$  and  $z$  axes which are parallel and perpendicular to the airplane thrust line. The



combined weight and inertia load on any element has a component along the  $z$  axis of the following magnitude:

$$nw = w \cos \theta + w \frac{a_z}{g}$$

or

$$n = \cos \theta + \frac{a_z}{g} \quad (3.23)$$

From a summation of all forces along the  $z$  axis,

$$L = W \left( \cos \theta + \frac{a_z}{g} \right) \quad (3.24)$$

By combining Eqs. 3.23 and 3.24,

$$L = Wn$$

or

$$n = \frac{L}{W}$$

which corresponds with the value used in Eq. 3.21 for a level attitude of the airplane.

The thrust load factor for the condition shown in Fig. 3.25 is also similar to that obtained for the airplane in level attitude. Since the thrust and drag forces must be in equilibrium with the components of weight and inertia forces along the  $x$  axis, the thrust load factor is obtained as follows:

$$n_x W = \frac{W}{g} a_x - W \sin \theta = T - D$$

or

$$n_x = \frac{T - D}{W}$$

This value is the same as that obtained in Eq. 3.22 for a level attitude of the airplane.

In the case of the airplane landing as shown in Fig. 3.26, the landing load factor is defined as the vertical ground reaction divided by the airplane weight. The load factor in the horizontal direction is similarly defined as the horizontal ground reaction divided by the airplane weight.

$$n = \frac{R_z}{W} \quad (3.25)$$

$$n_x = \frac{R_x}{W} \quad (3.26)$$

In the airplane analysis it is necessary to obtain the components of the load factor along axes parallel and perpendicular to the propeller thrust line. Aerodynamic forces, however, are usually first obtained as

lift and drag forces perpendicular and parallel to the direction of flight. If load factors are first obtained along lift and drag axes, they may be resolved into components along other axes, in the same manner that forces are resolved into components.

The force acting on any weight  $w$  is  $wn$ , and the component of this force along any axis at an angle  $\theta$  to the force is  $wn \cos \theta$ . The component of the load factor is then  $n \cos \theta$ .

As a general definition, the load factor  $n_i$  along any axis  $i$  is such that the product of the load factor and the weight of an element is equal to the sum of the components of the weight and inertia forces along that axis. The weight and inertia forces are always in equilibrium with the external forces acting on the airplane, and the sum of the components of the weight and inertia forces along any axis must be equal and opposite to the sum of the components of the external forces along the axis  $\Sigma F_i$ . The load factor is therefore defined as follows:

$$n_i = \frac{-\Sigma F_i}{W} \quad (3.27)$$

where  $\Sigma F_i$  includes all forces except weight and inertia forces.

In many airplane loading conditions, only translational acceleration is considered even though the airplane has some rotational motion. In Example 2, Art. 3.2, for example, the airplane has a rotational velocity during the pull-out from a dive. The inertia forces all act through the center of curvature of the flight path, although they are assumed to be parallel for various elements of the airplane. It is obvious that the inertia force for a mass near the nose of the airplane has a forward component, and the inertia force for a mass near the tail has an aft component. Since the radius of the flight path circle is large compared with the length of the airplane, these components are negligible.

The maximum loads which an airplane may be expected to encounter at any time in service are designated as *limit loads* or *applied loads*. The load factors associated with these loads are known as *limit load factors*, or *applied load factors*. For loads which are under the control of the pilot, flight restrictions are used so that the limit load factor is never exceeded. All parts of the airplane are designed so that they are not stressed beyond the yield point at the limit load factor. It is also necessary to use a safety factor in the structural design of airplanes.

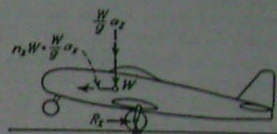


FIG. 3.26.



In most cases a safety factor of 1.5 based on the ultimate strength of the members is used, although higher factors are used for castings, fittings, and joints. The load factor obtained by multiplying the applied or limit load factor by the factor of safety is known as the *design load factor*, or *ultimate load factor*. The airplane structure must carry the ultimate or design loads without collapsing, even though the members may acquire permanent deformation under these loads.

The load factors for which airplanes must be designed are discussed in detail in a later chapter. The load factors are usually specified by the purchaser, or by the government licensing agency. The load factors depend on the purpose for which the airplane is intended and on the aerodynamic characteristics of the airplane. Military airplanes such as fighters or dive bombers are designed so that the airplane is strong enough to withstand any load factor that the pilot can withstand. Experience has shown that some pilots can withstand load factors of 7 or 8 before losing consciousness or "blacking out"; therefore a fighter or dive-bomber airplane would be designed for a limit or applied load factor of about 8, or a design or ultimate load factor of about 12. Large transport airplanes are never intentionally maneuvered to exceed a load factor much greater than 1. The greatest flight loads encountered by such airplanes occur as a result of air "gusts," or ascending air currents. The gust load factors depend on the gust velocity, the airplane velocity, and the airplane wing loading. Other types of airplanes are designed for load factors between those for the large transports and those for the fighters.

Landing load factors also depend on the function of the airplane. Large transport airplanes are expected to be landed on paved runways by experienced pilots, and need not be designed for large landing load factors. Training airplanes which may be operated from rough fields by inexperienced pilots must be designed for rather large landing load factors. The required load factor for any airplane is specified on the basis of the type of airplane, the airplane weight, and the wing loading.

**3.4. Load Factors for Angular Acceleration.** In the design of private or commercial airplanes it is seldom necessary to make an extensive analysis of the inertia forces resulting from rotational accelerations, since the loads in these conditions will determine the design for only a very few of the structural members, and conservative approximations do not result in excessive structural weight. In most of the loading conditions for military airplanes the angular accelerations are also neglected. In the design of Navy airplanes, which are often subjected to unusual loading conditions such as catapulting, arrested landings, barrier crashes, or seaplane landings, it is necessary to calculate the

moment of inertia of the airplane with reasonable accuracy and to consider various conditions of angular acceleration.

In the two-wheel landing analyzed in Example 1, Art. 3.2, the landing load factor for the airplane was 4.5; yet the total load down on the 400-lb turret was 3,850 lb. Of this load, 1,800 lb results from the weight and the translational acceleration, and 2,050 lb results from the angular acceleration of the airplane. It is obviously very important that the angular acceleration be considered when designing the supporting structure for the turret, but it is doubtful that much of the remaining airplane structure would be designed from this loading condition. The aerodynamic loads on the tail surfaces during various flight conditions will be much larger than the inertia loads during landing, and consequently most of the empennage and fuselage structure will be designed for the flight condition. For airplanes with the tail-wheel type of landing gear, the angular acceleration during landing is much smaller than that computed for the airplane with the nose-wheel type of landing gear.

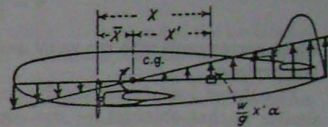


FIG. 3.27.

The vertical force  $F$  acting on an element of weight  $w$  when the airplane has a pitching acceleration  $\alpha$  consists of one force equal to the airplane load factor times the weight and of another force proportional to the distance of the element from the center of gravity, as shown in Fig. 3.27.

$$F = nw - \frac{w}{g} x' \alpha \quad (3.28)$$

The distance  $x'$  is measured from the center of gravity and must be multiplied by the weight. In the weight and balance calculations for the airplane, the terms  $w\bar{x}$  are calculated for all items of weight, where  $\bar{x}$  is measured from some reference plane a distance  $\bar{x}$  forward of the center of gravity. From Fig. 3.27,

$$x' = x - \bar{x}$$

substituting this value of  $x'$  in Eq. 3.28,

$$\left. \begin{aligned} F &= nw - \frac{\alpha}{g} w(x - \bar{x}) \\ F &= n'w - \frac{\alpha}{g} wx \end{aligned} \right\} \quad (3.29)$$

where

$$n' = n + \frac{\alpha \bar{x}}{g}$$



In most cases a safety factor of 1.5 based on the ultimate strength of the members is used, although higher factors are used for castings, fittings, and joints. The load factor obtained by multiplying the applied or limit load factor by the factor of safety is known as the *design load factor*, or *ultimate load factor*. The airplane structure must carry the ultimate or design loads without collapsing, even though the members may acquire permanent deformation under these loads.

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The vertical force  $F$  acting on an element of weight  $w$  when the airplane has a pitching acceleration  $\alpha$  consists of one force equal to the airplane load factor times the weight and of another force proportional to the distance of the element from the center of gravity, as shown in Fig. 3.27.

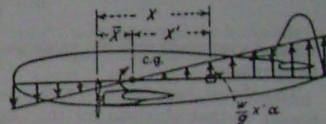


FIG. 3.27.

$$F = nw - \frac{w}{g} x' \alpha \quad (3.28)$$

The distance  $x'$  is measured from the center of gravity and must be multiplied by the weight. In the weight and balance calculations for the airplane, the terms  $w\bar{x}$  are calculated for all items of weight, where  $\bar{x}$  is measured from some reference plane a distance  $\bar{x}$  forward of the center of gravity. From Fig. 3.27,

$$x' = \bar{x} - \bar{x}$$

substituting this value of  $x'$  in Eq. 3.28,

$$\left. \begin{aligned} F &= nw - \frac{\alpha}{g} w(\bar{x} - \bar{x}) \\ F &= n'w - \frac{\alpha}{g} w\bar{x} \end{aligned} \right\} \quad (3.29)$$

where

$$n' = n + \frac{\alpha \bar{x}}{g}$$



The last term in Eq. 3.29 is easily evaluated by multiplying the tabulated values of  $wz$  by the ratio  $\alpha/g$ . The load factor  $n'$  represents the load factor for an element of weight located at the reference plane.

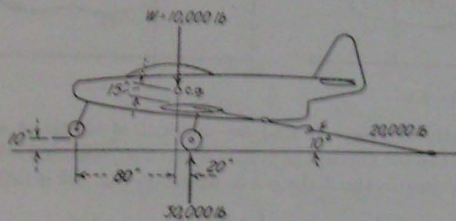
### PROBLEMS

3.5. Repeat Example 1, Art. 3.2, assuming that the airplane lands on a hard runway and that there is no horizontal ground reaction on the wheels. Find the landing load factor for a point at the center of gravity and also for a point a distance  $x$  from the center of gravity.

3.6. Assume the airplane of Example 1, Art. 3.2, to land in such a way that the nose wheel strikes the runway when the main wheels are 12 in. above the runway. The reaction on the nose wheel is vertical and has a constant value of 60,000 lb until the main wheels touch. Find the load factor for a point at the center of gravity, a point 200 in. forward of the center of gravity, and a point 500 in. aft of the center of gravity. Find the distance the nose-wheel shock strut is compressed at the time the main wheels touch if the vertical velocity of descent is 12 ft/sec before landing.

3.7. An airplane is making a horizontal turn with a radius of 1,000 ft, and with no change in altitude. Find the angle of bank and the load factor for a speed of (a) 200 mph, (b) 300 mph, and (c) 400 mph. Find the loads on the wing and tail if the dimensions of the airplane are shown in Fig. 3.18.

3.8. The airplane shown is making an arrested landing on a carrier deck. Find the load factors  $n$  and  $n_x$ , perpendicular and parallel to the deck, for a point at the center of gravity, a point 200 in. aft of the center of gravity, and a point



PROB. 3.8.

100 in. forward of the center of gravity. Find the relative vertical velocity with which the nose wheel strikes the deck if the vertical velocity of the center of gravity is 12 ft/sec and the angular velocity is 0.5 rad/sec counterclockwise for the position shown. The radius of gyration for the mass of the airplane about the center of gravity is 60 in. Assume no change in the dimensions or loads shown.

### CHAPTER 4

#### MOMENTS OF INERTIA, MOHR'S CIRCLE

4.1. Centroids. The force of gravity acting on any body is the resultant of a group of parallel forces acting on all elements of the body.

The magnitude of the resultant of several parallel forces is equal to the algebraic sum of the forces, and the position of the resultant is such that it has a moment about any axis equal to the sum of the moments of the component forces. The resultant gravity force on a body is its weight  $W$ , which must be equal to the sum of the weights  $w$ , of all elements of the body. If the forces of gravity act parallel to the  $z$  axis, as shown in Fig. 4.1, the moments of all forces about the  $x$  and  $y$  axes must be equal to the moment of the resultant.

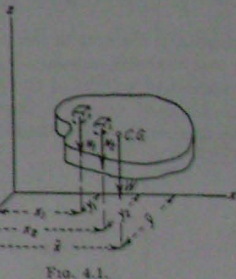


FIG. 4.1.

$$W\bar{x} = x_1w_1 + x_2w_2 + \dots = \sum xw \quad \text{or} \quad \int x dW \quad (4.1)$$

$$W\bar{y} = y_1w_1 + y_2w_2 + \dots = \sum yw \quad \text{or} \quad \int y dW \quad (4.2)$$

If the body and the axes are rotated so that the forces are parallel to one of the other axes, a third moment equation can be used.

$$W\bar{z} = z_1w_1 + z_2w_2 + \dots = \sum zw \quad \text{or} \quad \int z dW \quad (4.3)$$

The three coordinates  $\bar{x}$ ,  $\bar{y}$ , and  $\bar{z}$  of the center of gravity (c.g.) may be obtained from Eqs. 4.1 to 4.3.

$$\bar{x} = \frac{\sum xw}{W} \quad \text{or} \quad \frac{\int x dW}{W} \quad (4.4)$$

$$\bar{y} = \frac{\sum yw}{W} \quad \text{or} \quad \frac{\int y dW}{W} \quad (4.5)$$

$$\bar{z} = \frac{\sum zw}{W} \quad \text{or} \quad \frac{\int z dW}{W} \quad (4.6)$$



The last term in Eq. 3.29 is easily evaluated by multiplying the tabulated values of  $w\bar{x}$  by the ratio  $\alpha/g$ . The load factor  $n'$  represents the load factor for an element of weight located at the reference plane.

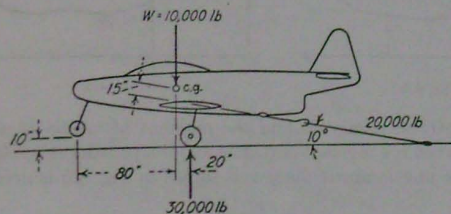
## PROBLEMS

3.5. Repeat Example 1, Art. 3.2, assuming that the airplane lands on a hard runway and that there is no horizontal ground reaction on the wheels. Find the landing load factor for a point at the center of gravity and also for a point a distance  $x$  from the center of gravity.

3.6. Assume the airplane of Example 1, Art. 3.2, to land in such a way that the nose wheel strikes the runway when the main wheels are 12 in. above the runway. The reaction on the nose wheel is vertical and has a constant value of 60,000 lb until the main wheels touch. Find the load factor for a point at the center of gravity, a point 200 in. forward of the center of gravity, and a point 500 in. aft of the center of gravity. Find the distance the nose-wheel shock strut is compressed at the time the main wheels touch if the vertical velocity of descent is 12 ft/sec before landing.

3.7. An airplane is making a horizontal turn with a radius of 1,000 ft, and with no change in altitude. Find the angle of bank and the load factor for a speed of (a) 200 mph, (b) 300 mph, and (c) 400 mph. Find the loads on the wing and tail if the dimensions of the airplane are shown in Fig. 3.18.

3.8. The airplane shown is making an arrested landing on a carrier deck. Find the load factors  $n$  and  $n_z$ , perpendicular and parallel to the deck, for a point at the center of gravity, a point 200 in. aft of the center of gravity, and a point



PROB. 3.8.

100 in. forward of the center of gravity. Find the relative vertical velocity with which the nose wheel strikes the deck if the vertical velocity of the center of gravity is 12 ft/sec and the angular velocity is 0.5 rad/sec counterclockwise for the position shown. The radius of gyration for the mass of the airplane about the center of gravity is 60 in. Assume no change in the dimensions or loads shown.

## CHAPTER 4

## MOMENTS OF INERTIA, MOHR'S CIRCLE

4.1. Centroids. The force of gravity acting on any body is the resultant of a group of parallel forces acting on all elements of the body.

The magnitude of the resultant of several parallel forces is equal to the algebraic sum of the forces, and the position of the resultant is such that it has a moment about any axis equal to the sum of the moments of the component forces. The resultant gravity force on a body is its weight  $W$ , which must be equal to the sum of the weights  $w_i$  of all elements of the body. If the forces of gravity act parallel to the  $z$  axis, as shown in Fig. 4.1, the moments of all forces about the  $x$  and  $y$  axes must be equal to the moment of the resultant.

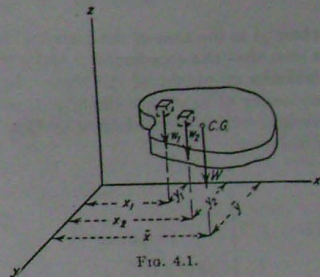


FIG. 4.1.

$$W\bar{x} = x_1w_1 + x_2w_2 + \dots = \Sigma xw \quad \text{or} \quad \int x dW \quad (4.1)$$

$$W\bar{y} = y_1w_1 + y_2w_2 + \dots = \Sigma yw \quad \text{or} \quad \int y dW \quad (4.2)$$

If the body and the axes are rotated so that the forces are parallel to one of the other axes, a third moment equation can be used.

$$W\bar{z} = z_1w_1 + z_2w_2 + \dots = \Sigma zw \quad \text{or} \quad \int z dW \quad (4.3)$$

The three coordinates  $\bar{x}$ ,  $\bar{y}$ , and  $\bar{z}$  of the center of gravity (c.g.) may be obtained from Eqs. 4.1 to 4.3.

$$\bar{x} = \frac{\Sigma xw}{W} \quad \text{or} \quad \frac{\int x dW}{W} \quad (4.4)$$

$$\bar{y} = \frac{\Sigma yw}{W} \quad \text{or} \quad \frac{\int y dW}{W} \quad (4.5)$$

$$\bar{z} = \frac{\Sigma zw}{W} \quad \text{or} \quad \frac{\int z dW}{W} \quad (4.6)$$



The summations or integrals for Eqs. 4.4 to 4.6 must include all elements of the body. In many engineering problems the weights and coordinates of the various items are known, and the center of gravity is obtained by a summation procedure, rather than by an integration procedure.

In the case of a plate of uniform thickness and density which lies in the  $xy$  plane, as shown in Fig. 4.2, the coordinates of the center of gravity are

$$\bar{x} = \frac{\int x dW}{W} = \frac{w \int x dA}{wA} = \frac{\int x dA}{A} \quad (4.7)$$

$$\bar{y} = \frac{\int y dW}{W} = \frac{w \int y dA}{wA} = \frac{\int y dA}{A} \quad (4.8)$$

where  $A$  is the area of the plate and  $w$  is the weight per unit area. It is seen that the coordinates  $\bar{x}$  and  $\bar{y}$  will be the same regardless of the thickness or weight of the plate. In many engineering problems the properties of areas are important, and the point in the area having coordinates  $\bar{x}$  and  $\bar{y}$  as defined by Eqs. 4.7 and 4.8 is called the centroid of the area.

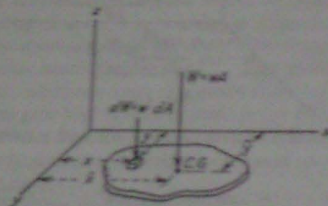


FIG. 4.2.

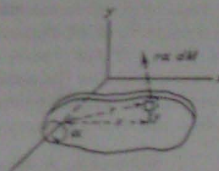


FIG. 4.3.

**4.2. Moment of Inertia.** In considering inertia forces on rotating masses, it was found that the inertia forces on the elements of mass had a moment about the axis of rotation of

$$L_x = a \int r^2 dM$$

as shown in Fig. 4.3. The term under the integral sign is defined as the moment of inertia of the mass about the  $x$  axis.

$$I_x = \int r^2 dM \quad (4.9)$$

Since the  $x$  and  $y$  coordinates of the elements are easier to tabulate than the radius, it is frequently convenient to use the relation

$$r^2 = x^2 + y^2$$

or

$$I_x = \int x^2 dM + \int y^2 dM \quad (4.10)$$

An area has no mass, and consequently no inertia, but it is customary to designate the following properties of an area as the moments of inertia of the area since they are similar to the moments of inertia of masses.

$$I_x = \int y^2 dA \quad (4.11)$$

$$I_y = \int x^2 dA \quad (4.12)$$

The coordinates are shown in Fig. 4.4. The polar moment of inertia of an area is defined as follows:

$$I_o = \int r^2 dA \quad (4.13)$$

From the relationship used in Eq. 4.10,

$$\left. \begin{aligned} r^2 &= x^2 + y^2 \\ I_o &= \int x^2 dA + \int y^2 dA = I_y + I_x \end{aligned} \right\} \quad (4.14)$$

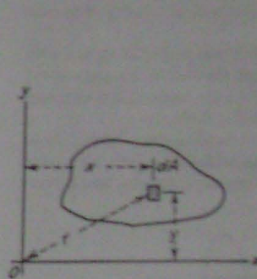


FIG. 4.4.

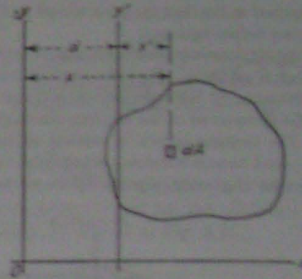


FIG. 4.5.

It is frequently necessary to find the moment of inertia of an area about an axis when the moment of inertia about a parallel axis is known. The moment of inertia about the  $y$  axis shown in Fig. 4.5 is defined as follows:

$$I_y = \int x^2 dA \quad (4.15)$$

substituting the relation  $x = d + x'$  in Eq. 4.15,

$$\begin{aligned} I_y &= \int (d + x')^2 dA \\ &= d^2 \int dA + 2d \int x' dA + \int x'^2 dA \end{aligned}$$

or

$$I_y = Ad^2 + 2x'Ad + I_y' \quad (4.16)$$

where  $x'$  represents the distance of the centroid of the area from the  $y'$  axis, as defined in Eq. 4.7,  $I_y'$  represents the moment of inertia of the

area about the  $y'$  axis, and  $A$  represents the total area. Equation 4.16 is simplified when the  $y'$  axis is through the centroid of the area, as shown in Fig. 4.6.

$$I_{y'} = Ad^2 + I_x \quad (4.17)$$

The term  $I_x$  represents the moment of inertia of the area about a centroidal axis.

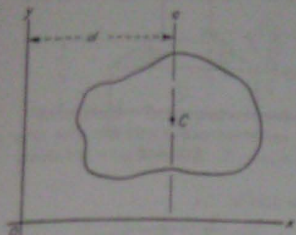


FIG. 4.6.

The moment of inertia of a mass may be transferred to a parallel axis by a similar procedure to that used for the moment of inertia of an area. For the mass shown in Fig. 4.7, the following relations apply where the centroidal axis  $C$  lies in the  $xz$  plane.

$$\begin{aligned} I_x &= \int r^2 dM = \int (x^2 + y^2) dM \\ &= \int [(d + x')^2 + y^2] dM \\ &= \int [d^2 + 2dx' + x'^2 + y^2] dM \end{aligned}$$

substituting  $r_1^2 = x'^2 + y^2$ ,

$$I_x = d^2 \int dM + 2d \int x' dM + \int r_1^2 dM$$

Since  $x'$  is measured from the centroidal axis, the second integral is zero. The last integral represents the moment of inertia about the centroidal axis.

$$I_x = Md^2 + I_x \quad (4.18)$$

Equations 4.17 and 4.18 can be used either to find the moment of inertia about any axis when the moment of inertia about a parallel axis through the centroid is known or to find the moment of inertia about the centroidal axis when the moment of inertia about any other parallel axis is known. In transferring moments of inertia between two axes, neither

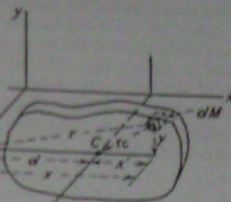


FIG. 4.7.

of which is through the centroid, it is necessary first to find the moment of inertia about the centroidal axis, then to transfer this to the desired axis, by using Eq. 4.17 or Eq. 4.18 two times. It is seen that the moment of inertia is always a positive quantity, and that the moment of inertia about a centroidal axis is always smaller than that about any other parallel axis. If Eq. 4.16 is used, it is necessary to use the proper sign for the term  $x'$ . All terms in Eqs. 4.17 and 4.18 are always positive.

The radius of gyration  $\rho$  of a body is the distance from the inertia axis that the entire mass would be concentrated in order to give the same moment of inertia. Equating the moment of inertia of the concentrated mass to that for the body,

$$\rho^2 M = I$$

or

$$\rho = \sqrt{\frac{I}{M}} \quad (4.19)$$

It is seen that the point where the mass is assumed to be concentrated is not the same as the center of gravity, except for the case where  $I_x$  in Eq. 4.18 is zero. The point at which the mass is assumed to be concentrated is also different for each inertia axis chosen.

The radius of gyration of an area is defined as the distance from the inertia axis to the point where the area would be concentrated in order to produce the same moment of inertia.

$$\rho^2 A = I$$

or

$$\rho = \sqrt{\frac{I}{A}} \quad (4.20)$$

The moment of inertia of an area is obtained as the product of an area and the square of a distance and is usually expressed in units of inches to the fourth power. The moments of inertia for the common areas shown

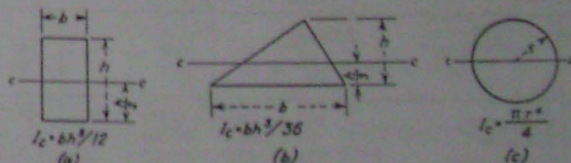


FIG. 4.8.

in Fig. 4.8 should be memorized, since they are frequently used. Moments of inertia for other areas may be found by integration or from engineering handbooks.





**Example 1.** Find the center of gravity of the airplane shown in Fig. 4.9(a). The various items of weight and the coordinates of their individual centers of gravity are shown in Table 4.1. It is customary to take reference axes in the directions shown in Fig. 4.9(b) with the  $x$  axis parallel to the thrust line and the  $z$  axis vertical. While the  $z$  axis is at the wing leading edge in this problem, it may be taken through the propeller or through some other convenient reference point.

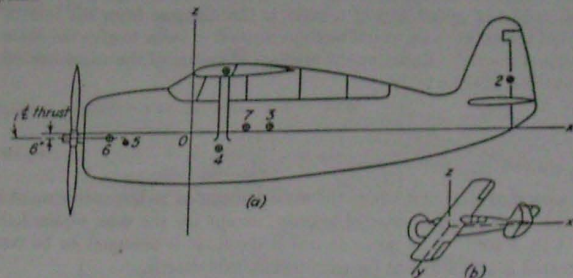


Fig. 4.9.

**Solution.** The  $y$  coordinate of the center of gravity is in the plane of symmetry of the airplane. The coordinates  $\bar{x}$  and  $\bar{z}$  are obtained from Eqs. 4.4 and 4.6. The terms  $W$ ,  $\Sigma wx$ , and  $\Sigma wz$  are obtained by totaling columns (3), (5), and (7) of Table 4.1.

$$\bar{x} = \frac{\Sigma wx}{W} = \frac{50,723}{4,243} = 12.16$$

$$\bar{z} = \frac{\Sigma wz}{W} = \frac{26,109}{4,243} = 6.2$$

TABLE 4.1

| No. (1) | Item (2)                  | Weight $W$ (3) | $x$ (4) | $wx$ (5) | $z$ (6) | $wz$ (7) |
|---------|---------------------------|----------------|---------|----------|---------|----------|
| 1       | Wing group.....           | 697            | 22.6    | +15,781  | 40.9    | +28,574  |
| 2       | Tail group.....           | 156            | 198.0   | 30,904   | 33.1    | 5,171    |
| 3       | Fuselage group.....       | 792            | 49.8    | 39,430   | 3.9     | 3,092    |
| 4       | Landing gear (up).....    | 380            | 19.2    | 7,297    | -11.7   | -4,429   |
| 5       | Engine section group..... | 160            | -38.6   | -6,179   | -7.1    | -1,138   |
| 6       | Power plant.....          | 1,302          | -48.8   | -63,674  | -6.0    | -7,782   |
| 7       | Fixed equipment.....      | 756            | 35.9    | 27,164   | 3.5     | 2,621    |
|         | Total weight empty.....   | 4,243          | .....   | 50,723   | .....   | 26,109   |

**Example 2.** Find the centroid and the moment of inertia about a horizontal axis through the centroid of the area shown in Fig. 4.10. Find the radii of gyration about axis  $xx$  and about axis  $cc$ .

**Solution.** The area is divided into rectangles and triangles, as shown. The areas of the individual parts are tabulated in column (2) of Table 4.2. The  $y$  coordinates of the centroids of the elements are tabulated in column (3), and the moments of the areas  $Ay$  tabulated in column (4). The centroid of the total area is now obtained by dividing the summation of the terms in column (4) by the summation of the terms in column (2).

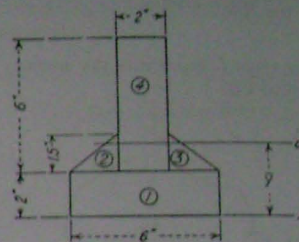


Fig. 4.10.

$$\bar{y} = \frac{\Sigma Ay}{A} = \frac{79.5}{27} = 2.94 \text{ in.}$$

TABLE 4.2

| Element (1) | $A$ (2) | $y$ (3) | $Ay$ (4) | $Ay^2$ (5) | $I_0$ (6) |
|-------------|---------|---------|----------|------------|-----------|
| 1           | 12      | 1       | 12       | 12         | 4.0       |
| 2           | 1.5     | 2.5     | 3.75     | 9.4        | 0.2       |
| 3           | 1.5     | 2.5     | 3.75     | 9.4        | 0.2       |
| 4           | 12      | 5       | 60       | 300        | 36.0      |
| Total...    | 27.0    |         | 79.5     | 330.8      | 40.4      |

The moment of inertia of the total area about the  $x$  axis will be obtained as the sum of the moments of inertia of the elements about this axis. In finding the moment of inertia of any element about the  $x$  axis, Eq. 4.17 may be written as follows:

$$I_x = Ay^2 + I_0$$

where  $I_x$  is the moment of inertia of the element of area  $A$  about the  $x$  axis,  $y$  is the distance from the centroid of the element to the  $x$  axis, and  $I_0$  is the moment of inertia of the element about its own centroid. The terms  $Ay^2$  for all the elements are obtained in column (5) as the product of terms in columns (3) and (4). The values of  $I_0$  are obtained from the equations shown in Fig. 4.8. The moment of inertia of the entire area about the  $x$  axis is equal to the sum of all terms in columns (5) and (6).

$$I_x = 330.8 + 40.4 = 371.2 \text{ in.}^4$$



This moment of inertia may be transferred to the centroid of the entire area by using Eq. 4.17 as follows:

$$I_x = I_x - (\Sigma A)\bar{y}^2$$

where  $\Sigma A$  represents the total area and  $\bar{y}$  represents the distance from the  $x$  axis to the centroid of the total area.

$$I_x = 371.2 - 27.0(2.94)^2 = 138.0 \text{ in.}^4$$

The radii of gyration may now be obtained as defined in Eq. 4.20.

$$\rho_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{138.0}{27.0}} = 3.71 \text{ in.}$$

$$\rho_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{138.0}{27.0}} = 2.26 \text{ in.}$$

**Example 3.** In a metal stressed-skin airplane wing, the sheet-metal covering acts with the supporting spanwise spars and stringers to form a beam which resists the wing bending. Figure 4.11(a) shows a cross section of a typical wing

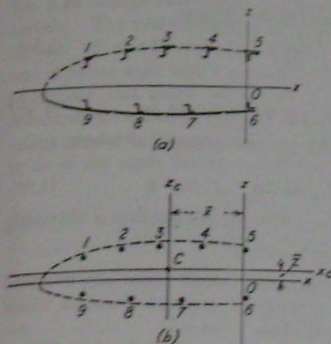


FIG. 4.11.

which has only one spar. The spar has a vertical web and extruded angle sections riveted to the spar web and to the skin. The stringers are extruded Z sections which are riveted to the skin. The upper surface of the wing is in compression, and the sheet-metal skin buckles between the stringers and is ineffective in carrying load. The skin is riveted to the stringers at frequent intervals, and a narrow strip of skin adjacent to each stringer is prevented from buckling and acts with the stringer in carrying compressive load. The effective width of skin acting with each stringer is usually about thirty times the skin thickness, but will be computed by more accurate equations in a subsequent chapter. On the under side of the wing the entire width of the skin is effective in resisting tension. It is usually sufficiently accurate to assume the area of each stringer and its effective skin to be concentrated at the centroid of its area in computing the moment of inertia of the area. The wing cross section would then be represented by the nine elements of area shown in Fig. 4.11(b). The moment of inertia of each element about its own centroid is neglected. In this particular wing, the skin and stringers to the right of the spar are very light and are assumed to be nonstructural.

The moment of inertia of the area shown in Fig. 4.11(b) will be obtained about horizontal and vertical axes through the centroid of the total area. The areas and coordinates of the elements are given in columns (2), (3), and (6) of Table 4.3.

**Solution.** This problem is solved by the method used for Example 2, except

that the column for  $I_x$  is omitted. Table 4.3 shows the calculations for the moments of inertia about both the horizontal and vertical axes.

TABLE 4.3

| Element<br>(1) | A<br>(2) | $\bar{x}$<br>(3) | $A\bar{x}$<br>(4) | $A\bar{x}^2$<br>(5) | $\bar{y}$<br>(6) | $A\bar{y}$<br>(7) | $A\bar{y}^2$<br>(8) |
|----------------|----------|------------------|-------------------|---------------------|------------------|-------------------|---------------------|
| 1              | 0.358    | -34.5            | -12.34            | 426                 | +8.6             | 3.08              | 26.5                |
| 2              | 0.294    | -28.1            | -8.27             | 231                 | +9.6             | 1.96              | 18.8                |
| 3              | 0.395    | -19.9            | -7.85             | 156                 | +10.0            | 3.95              | 39.5                |
| 4              | 0.294    | -10.1            | -2.96             | 21                  | +9.6             | 1.96              | 18.8                |
| 5              | 1.615    | +0.5             | +0.81             | 0                   | 8.8              | 14.21             | 125.2               |
| 6              | 1.931    | +0.5             | +0.97             | 1                   | -5.7             | -11.02            | 62.8                |
| 7              | 0.752    | -10.1            | -7.60             | 77                  | -5.2             | -3.91             | 29.4                |
| 8              | 0.784    | -22.4            | -17.65            | 394                 | -4.3             | -3.37             | 14.5                |
| 9              | 0.892    | -34.7            | -30.92            | 1,074               | -2.4             | -2.14             | 5.1                 |
| Total.....     | 7.135    | .....            | -82.40            | 2,310               | .....            | 4.72              | 331.6               |

$$\bar{x} = \frac{-82.4}{7.135} = -11.56$$

$$I_{xx} = 2,310 - 7.135(11.56)^2 = 1,358 \text{ in.}^4$$

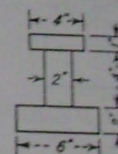
$$\bar{y} = \frac{4.72}{7.135} = 0.66$$

$$I_{yy} = 331.6 - 7.135(0.66)^2 = 328 \text{ in.}^4$$

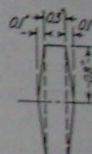
### PROBLEMS

4.1. Determine the moment of inertia of the area shown about a horizontal axis through the centroid.

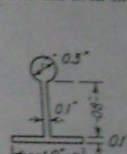
4.2. Determine the moments of inertia and radii of gyration about horizontal and vertical axes through the centroid of the area shown.



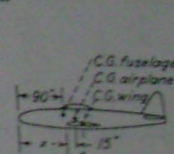
PROB. 4.1.



PROB. 4.2.



PROB. 4.3.



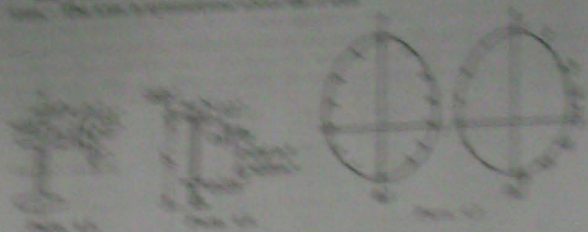
PROB. 4.4.

4.3. Find the moments of inertia and radii of gyration about horizontal and vertical axes through the centroid of the area shown.

4.4. For proper balance of the 10,000-lb airplane shown, the wing must be located so that the airplane center of gravity is 15 in. aft of the wing leading edge. The center of gravity of the 1,500-lb wing is 25 in. aft of the leading edge. The remaining 8,500 lb of weight is considered to be in the fuselage and has its center of gravity 90 in. aft of the nose, as shown. Find the distance  $x$  from the nose of the fuselage to the wing leading edge.

# MEASUREMENT OF SPIN ABOUT THE Z-AXIS

Let the wave function  $\psi$  be a function of the spin coordinate  $\sigma$ . The wave function  $\psi$  is a function of  $\sigma$  only.



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Table 41

| Wave | $\psi$                                       | $\sigma$             | Wave | $\psi$                                       | $\sigma$              |
|------|----------------------------------------------|----------------------|------|----------------------------------------------|-----------------------|
| 1    | $\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$ | $\frac{1}{\sqrt{2}}$ | 2    | $\psi = \frac{1}{\sqrt{2}}(\psi_1 - \psi_2)$ | $-\frac{1}{\sqrt{2}}$ |
| 3    | $\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$ | $\frac{1}{\sqrt{2}}$ | 4    | $\psi = \frac{1}{\sqrt{2}}(\psi_1 - \psi_2)$ | $-\frac{1}{\sqrt{2}}$ |

Let the wave function  $\psi$  be a function of the spin coordinate  $\sigma$ . The wave function  $\psi$  is a function of  $\sigma$  only. The wave function  $\psi$  is a function of  $\sigma$  only.

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \quad (4.11)$$

the wave function  $\psi$  is a function of  $\sigma$  only.

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \quad (4.12)$$

# MEASUREMENT OF SPIN ABOUT THE Z-AXIS

Let the wave function  $\psi$  be a function of the spin coordinate  $\sigma$ . The wave function  $\psi$  is a function of  $\sigma$  only.

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \quad (4.13)$$

The wave function  $\psi$  is a function of the spin coordinate  $\sigma$ . The wave function  $\psi$  is a function of  $\sigma$  only. The wave function  $\psi$  is a function of  $\sigma$  only.



$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \quad (4.14)$$

The wave function  $\psi$  is a function of the spin coordinate  $\sigma$ .

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \quad (4.15)$$

A similar expression may be derived for the wave function  $\psi$ .

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \quad (4.16)$$

Let the wave function  $\psi$  be a function of the spin coordinate  $\sigma$ .

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \quad (4.17)$$

Let the wave function  $\psi$  be a function of the spin coordinate  $\sigma$ . The wave function  $\psi$  is a function of  $\sigma$  only.

The wave function  $\psi$  is a function of the spin coordinate  $\sigma$ .

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \quad (4.18)$$

Let the wave function  $\psi$  be a function of the spin coordinate  $\sigma$ .

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \quad (4.19)$$

Let the wave function  $\psi$  be a function of the spin coordinate  $\sigma$ .

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \quad (4.20)$$

Let the wave function  $\psi$  be a function of the spin coordinate  $\sigma$ .

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \quad (4.21)$$

Let the wave function  $\psi$  be a function of the spin coordinate  $\sigma$ .

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \quad (4.22)$$



**4.4. Principal Axes.** The moment of inertia of any area about an inclined axis is a function of the angle  $\phi$ , as given in Eqs. 4.25 and 4.26. The angle  $\phi$  at which the moment of inertia  $I_\phi$  is a maximum or minimum is obtained from the derivative of Eq. 4.25 with respect to  $\phi$ .

$$\frac{dI_\phi}{d\phi} = -2I_{xy} \cos \phi \sin \phi - 2I_{xx} \cos 2\phi + 2I_{yy} \sin \phi \cos \phi$$

This derivative is zero when  $I_\phi$  is a maximum or minimum. Equating the derivative to zero and simplifying,

$$(I_{xx} - I_{yy}) \sin 2\phi = 2I_{xy} \cos 2\phi$$

$$\text{or} \quad \tan 2\phi = \frac{2I_{xy}}{I_{xx} - I_{yy}} \quad (4.29)$$

Since there are two angles under  $360^\circ$  which have the same tangent, Eq. 4.29 defines two values of the angle  $2\phi$ , which will be at  $180^\circ$  intervals. The two corresponding values of the angle  $\phi$  will be at  $90^\circ$  intervals. It can be shown that the value of  $I_\phi$  will be a maximum about one of these axes and a minimum about the other. These two perpendicular axes about which the moment of inertia is a maximum or minimum are called the *principal axes*.

The product of inertia about the inclined axes may be expressed in terms of the angle  $2\phi$ , by making use of the trigonometric relations

$$\sin 2\phi = 2 \sin \phi \cos \phi \quad (4.30)$$

$$\text{and} \quad \cos 2\phi = \cos^2 \phi - \sin^2 \phi \quad (4.31)$$

Substituting these values in Eq. 4.28, the following relation is obtained.

$$I_{\phi\phi} = I_{xx} \cos^2 2\phi + \frac{I_{xx} - I_{yy}}{2} \sin 2\phi \quad (4.32)$$

An important relation is obtained for the angle at which  $I_{\phi\phi}$  is zero. Substituting  $I_{\phi\phi} = 0$  in Eq. 4.32,

$$\tan 2\phi = \frac{2I_{xy}}{I_{xx} - I_{yy}}$$

This is identical to the expression defining the principal axes in Eq. 4.29. The product of inertia about the principal axes is therefore zero.

The moments of inertia about the principal axes may be obtained by substituting the value of  $\phi$  obtained from Eq. 4.29 into Eq. 4.25.

$$I_p = \frac{I_{xx} + I_{yy}}{2} + \sqrt{I_{xx}^2 + \left(\frac{I_{xx} - I_{yy}}{2}\right)^2} \quad (4.33)$$

and

$$I_q = \frac{I_{xx} + I_{yy}}{2} - \sqrt{I_{xx}^2 + \left(\frac{I_{xx} - I_{yy}}{2}\right)^2} \quad (4.34)$$

where  $I_p$  represents the maximum value of  $I_\phi$  and  $I_q$  represents the minimum value of  $I_\phi$ . These values are moments of inertia about perpendicular axes defined by Eq. 4.29.

**4.5. Product of Inertia.** The product of inertia of an area is evaluated by methods similar to those used in evaluating the moment of inertia. Products of inertia for various elements of the area are usually evaluated separately and then added in order to obtain the product of inertia for the entire area. When both  $x$  and  $y$  are positive or negative, the product of inertia is positive, but when one coordinate is positive and the other negative the product of inertia is negative. In the case of an area which is symmetrical with respect to the  $x$  axis, as shown in Fig. 4.13, each

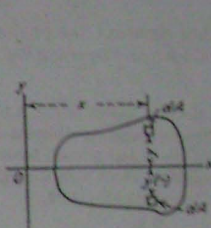


FIG. 4.13.

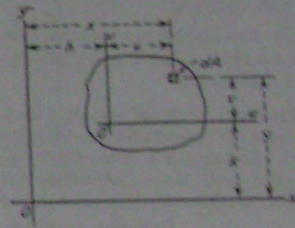


FIG. 4.14.

element of area  $dA$  in the first quadrant will have a corresponding area in the fourth quadrant with the same  $x$  coordinate but with the  $y$  coordinate changed in sign. The sum of the products of inertia for the two elements will be zero, and the integral of these terms for the entire area will be zero.

$$I_{xy} = \int xy \, dA = 0$$

The same relation is true if the area is symmetrical with respect to the  $y$  axis. Therefore, when either axis is an axis of symmetry, the product of inertia is zero and the axes are principal axes.

When the product of inertia of an area about one set of coordinate axes is known, it is possible to find the product of inertia about a set of parallel axes. For the area shown in Fig. 4.14, the product of inertia about the  $x$  and  $y$  axes is defined as

$$I_{xy} = \int xy \, dA$$

Substituting the values

$$\begin{aligned}x &= h + u \\y &= k + v\end{aligned}$$

the transfer theorem is obtained.

$$\begin{aligned}I_{xx} &= \int (h + u)^2 (k + v)^2 dA \\&= hk \int dA + k^2 \int v^2 dA + h^2 \int u^2 dA + \int uv dA \\I_{xx} &= hkA + k^2 I_{xx'} + h^2 I_{yy'} + I_{xy'}\end{aligned}\quad (4.35)$$

where  $h$  and  $k$  are the coordinates of the centroid of the area and  $I_{xx'}$  is the product of inertia of the area with respect to the  $u$  and  $v$  axes. If the  $u$  and  $v$  axes are through the centroid of the area, Eq. 4.35 becomes

$$I_{xx} = hkA + I_{xx'} \quad (4.36)$$

If the  $u$  and  $v$  axes are also principal axes of the area,  $I_{xy'} = 0$ , and Eq. 4.36 becomes

$$I_{xx} = hkA \quad (4.37)$$

For an area composed of several asymmetrical elements, the product of inertia may be obtained as the sum of the values found by using Eq. 4.37 for each element.

**Example 3.** Find the product of inertia for the area shown in Fig. 4.15.

**Solution.** The total area is divided into the three rectangular elements  $A$ ,  $B$ , and  $C$ . Rectangle  $A$  is symmetrical about both the  $x$  and  $y$  axes; hence the product of inertia is zero. Rectangle  $B$  is symmetrical about axes through its centroid; therefore the product of inertia may be found from Eq. 4.37.

$$I_{xx} = hA = -2 \times 5 \times 8 = -120 \text{ in.}^4$$

For rectangle  $C$ ,

$$I_{xx} = hA = 2 \times -5 \times 8 = -120 \text{ in.}^4$$

For the total area,

$$I_{xx} = 0 - 120 - 120 = -240 \text{ in.}^4$$

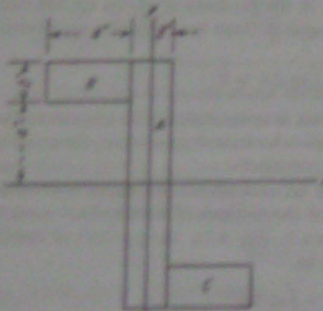


FIG. 4.15

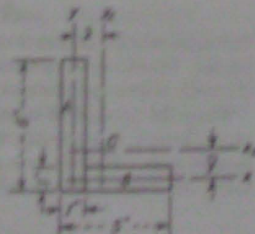


FIG. 4.16

**Example 2.** Find the product of inertia about horizontal and vertical axes through the centroid of the area shown in Fig. 4.16.

**Solution.** The total area is divided into the two rectangular elements  $A$  and  $B$  as shown. The  $x$  and  $y$  reference axes are chosen through the centroids of these rectangles. Since rectangle  $A$  is symmetrical about the  $y$  axis and rectangle  $B$  is symmetrical about the  $x$  axis,  $I_{xy} = 0$ . The centroid of the area is obtained as follows:

$$\bar{x} = \frac{6 \times 2.5}{10} = 1.5$$

$$\bar{y} = \frac{6 \times 2.5}{10} = 1.5$$

The product of inertia about the centroidal axes may now be obtained from Eq. 4.36

$$\begin{aligned}I_{xy} &= hA + I_{xy'} \\ &= 1.5 \times 1.5 \times 10 + I_{xy'}\end{aligned}$$

or

$$I_{xy'} = -15 \text{ in.}^4$$

**Example 3.** Find the product of inertia about horizontal and vertical axes through the centroid of the area shown in Fig. 4.17. The areas and coordinates of the elements are given in Table 4.5 and are reported in columns (2), (3), and (4) of Table 4.5.

TABLE 4.5

| Element<br>(1) | $A$<br>(2) | $x$<br>(3) | $y$<br>(4) | $Axy$<br>(5) |
|----------------|------------|------------|------------|--------------|
| 1              | 0.858      | -34.5      | +8.5       | -100.2       |
| 2              | 0.394      | -28.1      | +5.6       | -55.0        |
| 3              | 0.895      | -19.9      | +10.0      | -78.1        |
| 4              | 0.204      | -18.1      | +5.6       | -19.8        |
| 5              | 1.615      | +6.5       | +8.8       | 7.1          |
| 6              | 1.931      | +6.5       | -5.7       | -5.5         |
| 7              | 0.752      | -18.1      | -5.3       | 39.8         |
| 8              | 0.784      | -22.4      | -4.3       | 75.0         |
| 9              | 0.402      | -24.7      | -2.4       | 74.3         |
| Total          | 7.135      |            |            | -68.2        |

**Solution.** The product of inertia about the  $x$  and  $y$  axes is obtained as the summation of the terms  $Axy$ , obtained in column (5) of Table 4.5. The centroidal axes were found in Example 3, Art. 4.2, to have coordinates  $\bar{x} = -11.56$ ,  $\bar{y} = 0.66$ . From Eq. 4.36,

$$\begin{aligned}I_{xy} &= hA + I_{xy'} \\ &= -68.2 = -11.56 \times 0.66 \times 7.135 + I_{xy'}\end{aligned}$$

or

$$I_{xy'} = -13.8$$



4.6. Mohr's Circle for Moments of Inertia. The equations for the moments and products of inertia about inclined axes are difficult to remember. It is often convenient to use a semigraphic solution which is easier to remember and which is an aid in visualizing the relationship between the moments of inertia about various axes. If the values of  $I_{xy}$  from Eq. 4.32 are plotted against values of  $I_x$  obtained from Eq. 4.25 for corresponding values of  $\phi$ , the points will all fall on the circle shown in Fig. 4.17. The maximum and minimum values of the moments of inertia are represented by points P and Q.

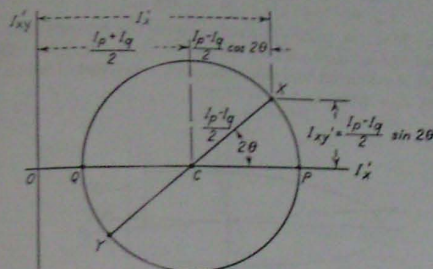


FIG. 4.17.

In order to prove that the circle shown in Fig. 4.17 represents the values given by Eqs. 4.25 and 4.32, the values of  $I_x$  and  $I_{xy}$  will be expressed in terms of the moments of inertia about the principal axes  $I_p$  and  $I_q$  and the angle  $\theta$  from the  $x'$  axis to the principal axis. If the  $x$  and  $y$  axes are principal axes, a substitution of the values  $I_{xy} = 0$ ,  $I_x = I_p$ ,  $I_y = I_q$ , and  $\phi = \theta$  into Eqs. 4.25 and 4.32 yields the following equations:

$$I_x = I_p \cos^2 \theta + I_q \sin^2 \theta \quad (4.38)$$

$$I_{xy} = \frac{I_p - I_q}{2} \sin 2\theta \quad (4.39)$$

The following trigonometric relations for double angles are used.

$$\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$$

$$\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta$$

Substituting these values in Eq. 4.38,

$$I_x = \frac{I_p + I_q}{2} + \frac{I_p - I_q}{2} \cos 2\theta \quad (4.40)$$

Equations 4.39 and 4.40 correspond to the coordinates  $I_{xy}$  and  $I_x$  which are computed from the geometry of the circle shown in Fig. 4.17. The

angle of inclination of the  $x'$  axis from the principal axis is called the angle  $2\theta$  measured between the corresponding points on the circle. If  $I_{xy}$  is measured as positive upward on the circle, a counterclockwise rotation of the  $x'$  axis corresponds to a counterclockwise rotation around the circle. Points at opposite ends of the diameter of the circle correspond to perpendicular inertia axes. The products of inertia about perpendicular axes are always equal numerically but opposite in sign, since rotation of the axes through  $90^\circ$  interchanges the numerical values of the coordinates  $x'$  and  $y'$  for any element of area and changes the sign of one coordinate.

**Example 1.** Find the moments of inertia about principal axes through the centroid of the area shown in Fig. 4.18. Find the moments and product of inertia about axes  $x_1$  and  $y_1$  and axes  $x_2$  and  $y_2$ .

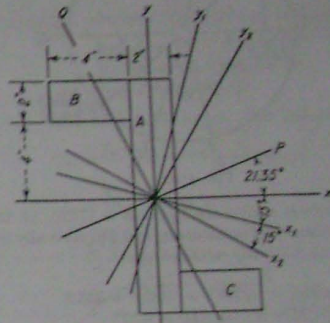


FIG. 4.18.

**Solution.** The moments of inertia about the  $x$  and  $y$  axes are obtained as follows:

$$I_x = \frac{2 \times 12^3}{12} + 2 \left( \frac{4 \times 2^3}{12} + 8 \times 5^2 \right) = 693.3 \text{ in.}^4$$

$$I_y = \frac{12 \times 4^3}{12} + 2 \left( \frac{2 \times 4^3}{12} + 8 \times 3^2 \right) = 173.3 \text{ in.}^4$$

From Example 1, Art. 4.5,

$$I_{xy} = -240 \text{ in.}^4$$

Mohr's circle for the moments and products of inertia about all inclined axes may now be plotted from these three values. The products of inertia  $I_{xy}$  are plotted against the moments of inertia  $I_x$ , as shown in Fig. 4.19. Point  $z$  in Fig. 4.19 has coordinates 693.3 and  $-240.0$ , as shown. If the  $x'$  axis is rotated



through  $90^\circ$ , it will coincide with the  $y$  axis, and the coordinates of point  $Y$  will be  $I_{x'} = 173.3$ ,  $I_{y'} = 240$ . The positive sign for  $I_{y'}$  results from the fact that after rotating the  $x'$  axis through  $90^\circ$  from the  $x$  axis the coordinate  $x'$  is positive up and the coordinate  $y'$  is positive to the left. The value of  $I_{x'y'}$  is numerically equal but opposite in sign to the value of  $I_{xy}$ . Points  $X$  and  $Y$  are at opposite ends of a diameter of the circle, since a rotation of the axes of  $90^\circ$  corresponds to an angle of  $180^\circ$  on the circle.

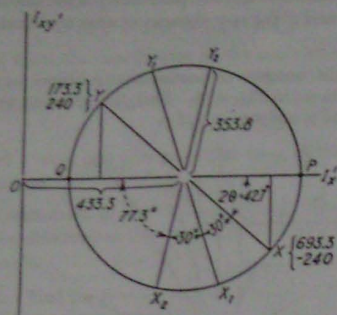


Fig. 4.19.

The center of the circle is a distance  $\frac{1}{2}(693.3 + 173.3)$ , or 433.3 from the origin. Point  $X$  is a distance 260 horizontally and 240 vertically from the center of the circle; therefore the following relations apply:

$$\text{Radius} = \sqrt{(240)^2 + (260)^2} = 353.8$$

$$2\theta = \arctan \frac{240}{260} = 42.7^\circ$$

or

$$\theta = 21.35^\circ$$

The principal axes are represented by points  $P$  and  $Q$  on the circle. The moments of inertia have maximum and minimum values, and the product of inertia is zero for these axes. The principal moments of inertia are equal to the distance from the origin to the center of the circle plus or minus the radius of the circle.

$$I_p = 433.3 + 353.8 = 787.1 \text{ in.}^4$$

$$I_q = 433.3 - 353.8 = 79.5 \text{ in.}^4$$

The  $P$  axis is counterclockwise from the  $x$  axis, at an angle  $\theta = 21.35^\circ$ . Similarly, since point  $Q$  on the circle is counterclockwise from point  $Y$ , the  $Q$  axis is counterclockwise from the  $y$  axis.

The moments and product of inertia about the  $x_1$  and  $y_1$  axes are obtained from the coordinates of points on the circle. Since the  $x_1$  and  $y_1$  axes are  $15^\circ$  clockwise from the  $x$  and  $y$  axes, the points  $X_1$  and  $Y_1$  on the circle will be  $30^\circ$

clockwise from points  $X$  and  $Y$ . The coordinates of points  $X_1$  and  $Y_1$  may be obtained from the geometry of the circle as follows:

$$I_{x_1} = 433.3 + 353.8 \cos 72.7^\circ = 538 \text{ in.}^4$$

$$I_{y_1} = 433.3 - 353.8 \cos 72.7^\circ = 328 \text{ in.}^4$$

$$I_{x_1 y_1} = -353.8 \sin 72.7^\circ = -338 \text{ in.}^4$$

The moments and product of inertia about the  $x_2$  and  $y_2$  axes are also obtained from the geometry of the circle.

$$I_{x_2} = 433.3 - 353.8 \cos 77.3^\circ = 355 \text{ in.}^4$$

$$I_{y_2} = 433.3 + 353.8 \cos 77.3^\circ = 511 \text{ in.}^4$$

$$I_{x_2 y_2} = -353.8 \sin 77.3^\circ = -345 \text{ in.}^4$$

**Example 2.** Find the principal axes through the centroid, and find the moments of inertia about these axes for the area shown in Fig. 4.20. The  $x$  and  $z$  axes are through the centroid, and the moments and product of inertia have the values  $I_x = 320$ ,  $I_z = 1,160$ , and  $I_{xz} = -120$ .

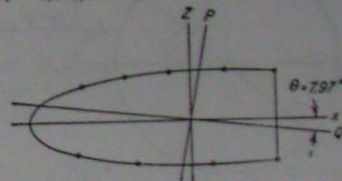


Fig. 4.20.

**Solution.** The coordinates of point  $X$  on the circle of Fig. 4.21 are 320 and -120. The coordinates of point  $Z$  are 1,160 and +120. These points are at opposite ends of the diameter and thus determine the circle. It is important to

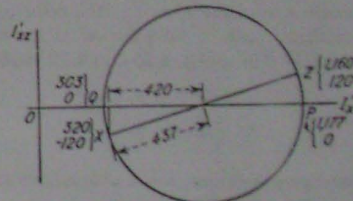


Fig. 4.21.

show  $I_{xz}$  with the correct sign at point  $X$  in order that the direction of the principal axes may be obtained correctly. The distance of the center of the circle from the origin of coordinates is  $\frac{1}{2}(320 + 1,160) = 740$ . From the geometry of the circle,

$$\text{Radius} = \sqrt{(420)^2 + (120)^2} = 437$$

$$\tan 2\theta = \frac{120}{420} \quad 2\theta = 15.94^\circ$$

or

$$\theta = 7.97^\circ$$



The principal moments of inertia are represented by points  $P$  and  $Q$  on the circle, at which the moments of inertia have maximum and minimum values and the product of inertia is zero.

$$I_x = 740 + 437 = 1,177 \text{ in.}^4$$

$$I_y = 740 - 437 = 303 \text{ in.}^4$$

Since point  $P$  is  $15.94^\circ$  clockwise from point  $Z$  on the circle, the  $P$  axis will be half this angle, or  $7.97^\circ$  clockwise from the  $z$  axis, as shown in Fig. 4.20.

**4.7. Mohr's Circle for Combined Stresses.** The relationship between normal stresses and shearing stresses on planes at various angles of inclination is similar to the relationship between moments and products of inertia about inclined axes. Most structural members are subjected simultaneously to normal and shearing stresses, and it is necessary to consider the combined effect of the stresses in order to design the members. The landing-gear strut shown in Fig. 4.22, for example, is subjected to bending stresses which produce tension in the direction of the strut, internal oil pressure which produces a circumferential tension, and torsion which produces shearing stresses on the horizontal and vertical planes. The maximum tensile stress does not occur on either the horizontal or vertical plane, but on a plane inclined at some angle to them. It can be shown that there are always two perpendicular planes on which the shearing stresses are zero.<sup>1</sup> These planes are called *principal planes*, and the stresses on these planes are called *principal stresses*.

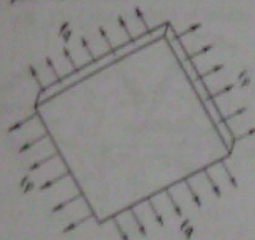


FIG. 4.23.

Any condition of two-dimensional stresses can be represented as shown in Fig. 4.23, in which the principal stresses  $f_p$  and  $f_q$  act on the perpendicular principal planes. The orientation of these planes depends on

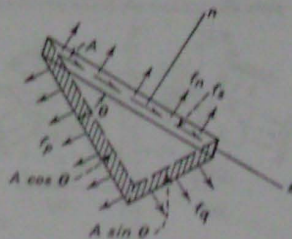


FIG. 4.24.

the condition of stress and will be found from known stress conditions. The normal and shearing stresses  $f_n$  and  $f_s$  can be found on a plane at an angle  $\theta$  to the principal planes, from the equations of statics. The stresses  $f$  are in pounds per square inch (psi) and must always be multiplied by the area in square inches in order to obtain the force. In Fig. 4.24, a small triangular element is shown, with principal planes forming two sides of the element and the inclined plane a third side. If the inclined plane has an area  $A$ , the sections of the principal planes have areas  $A \cos \theta$  and  $A \sin \theta$ , as shown. From a summation of forces along the  $n$  and  $s$  axes, which are perpendicular and parallel to the inclined plane, the following equations are obtained:

$$\Sigma F_n = f_n A - f_p A \cos^2 \theta - f_q A \sin^2 \theta = 0 \quad (4.41)$$

$$\Sigma F_s = f_s A - f_p A \cos \theta \sin \theta + f_q \sin \theta \cos \theta = 0 \quad (4.42)$$

Using the trigonometric relations for functions of double angles,

$$\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta$$

$$\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$$

$$\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$$

and dividing Eqs. 4.41 and 4.42 by  $A$ , the following equations are obtained:

$$f_n = \frac{f_x + f_y}{2} + \frac{f_x - f_y}{2} \cos 2\theta \quad (4.43)$$

$$f_s = \frac{f_x - f_y}{2} \sin 2\theta \quad (4.44)$$

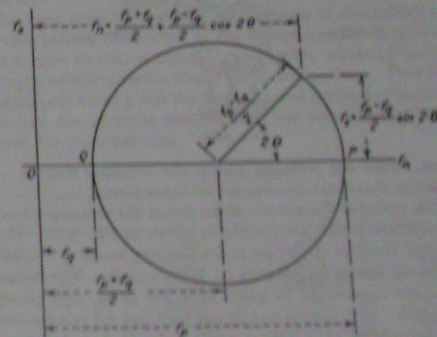


FIG. 4.25.

If values of  $f$ , and  $f_s$  are plotted for different values of the angle  $\theta$ , as shown in Fig. 4.25, all the points will lie on the circle. This construc-



tion, first used by Mohr, is similar to that used for finding moments of inertia about inclined axes.

Normal stresses are considered positive when tension and negative when compression. Compressive stresses are therefore shown to the left of the origin on Mohr's circle. In the following examples, shearing stresses will be considered positive when they tend to rotate the element clockwise and negative when they tend to rotate the element counterclockwise. Thus, on any two perpendicular planes the shearing stress on one plane would tend to rotate the element clockwise and would be measured upward on the circle, and the shearing stress on the other plane would be equal numerically but opposite in sign and would be measured downward. If this sign convention for shearing stresses is followed, a clockwise rotation of the planes of stress corresponds to a clockwise rotation on the circle. In some books the opposite sign convention for shearing stresses is used, and a clockwise rotation of the planes corresponds to a counterclockwise rotation on the circle.

**Example.** The small element shown in Fig. 4.26 represents the conditions of two-dimensional stress at a point in a structure. Find the normal and shearing stresses on planes inclined at an angle  $\theta$  with the vertical plane, for values of  $\theta$

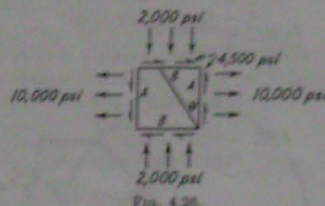


FIG. 4.26.

at  $30^\circ$  intervals. Find the principal planes and principal stresses. Find the planes of maximum shear and the stresses on these planes.

**Solution.** The values of the normal stress  $f_x$  and the shearing stress  $f_s$  on the horizontal and vertical planes are plotted as shown in Fig. 4.27. The vertical plane has a normal stress of 10,000 psi and a shearing stress of -4,500 psi, and these coordinates are shown for point A on Mohr's circle. The stresses on the horizontal plane are represented by point B on the circle, with coordinates of -2,000 and +4,500. The circle is now drawn with line AB as a diameter. The distance OC is  $\frac{1}{2}(-2,000 + 10,000) = 4,000$ . Points A and B have a horizontal distance of 6,000 and a vertical distance of 4,500 from the center of the circle.

$$\text{Radius} = \sqrt{(6,000)^2 + (4,500)^2} = 7,500$$

$$\tan 2\theta = \frac{4,500}{6,000} = 0.75$$

or

$$2\theta = 36.86^\circ \quad \text{and} \quad \theta = 18.43^\circ$$

The principal stresses are represented on the circle by points P and Q. The coordinates of these points are obtained by adding and subtracting the radius from distance OC.

$$f_p = 4,000 + 7,500 = 11,500 \text{ psi}$$

$$f_q = 4,000 - 7,500 = -3,500 \text{ psi}$$

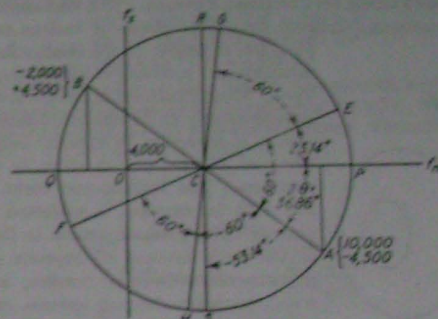


FIG. 4.27.

The point P on the circle is at an angle  $2\theta$  counterclockwise around the circle from point A. The principal plane P is therefore at the angle  $\theta = 18.43^\circ$  counterclockwise from the vertical plane A, as shown in Fig. 4.28(a). Similarly point Q is counterclockwise from the horizontal plane B, or perpendicular to plane P.

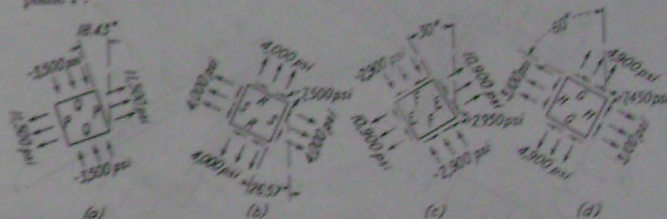


FIG. 4.28.

The planes of maximum shearing stress are always at  $45^\circ$  to the principal planes, regardless of the stress conditions. Points R and S at extremities of the vertical diameter of Mohr's circle represent the maximum shearing stresses. On the circle, these points are always  $90^\circ$  from the points at the extremities of the horizontal diameter, which represent the principal stresses. The normal



stresses on the two planes of maximum shear are always equal, since they are both equal to the distance  $OC$  on the circle diagram. The maximum shearing stresses are shown on an element in Fig. 4.28(b). Plane  $S$  is  $29.57^\circ$  clockwise from the vertical, since point  $S$  is twice this angle clockwise from point  $A$  on the circle. Plane  $S$  is also  $45^\circ$  clockwise from plane  $P$ , and plane  $R$  is  $45^\circ$  counterclockwise from plane  $P$ . The shearing stress on plane  $R$  is positive, tending to rotate the element clockwise. The shearing stress on plane  $S$  is negative, tending to rotate the element counterclockwise.

The planes  $E$  and  $F$  are  $30^\circ$  counterclockwise from planes  $A$  and  $B$ . Points  $E$  and  $F$  on the circle must be  $60^\circ$  counterclockwise from points  $A$  and  $B$ , respectively. The stresses on plane  $E$  are obtained by calculating the coordinates of point  $E$  on the circle.

$$f_s = 7,500 \sin 23.14^\circ = 2,950 \text{ psi}$$

$$f_n = 4,000 + 7,500 \cos 23.14^\circ = 10,900 \text{ psi}$$

The stresses on planes  $E$  and  $F$  are shown in Fig. 4.28(c) in the correct directions.

The stresses on plane  $G$ , which is  $60^\circ$  counterclockwise from the vertical plane  $A$ , are obtained as follows:

$$f_s = 7,500 \sin 83.14^\circ = 7,450 \text{ psi}$$

$$f_n = 4,000 + 7,500 \cos 83.14^\circ = 4,900 \text{ psi}$$

The stresses on planes  $H$ , which are perpendicular to plane  $G$ , are

$$f_s = -7,500 \sin 83.14^\circ = -7,450 \text{ psi}$$

$$f_n = 4,000 - 7,500 \cos 83.14^\circ = 3,100 \text{ psi}$$

These stresses are shown in Fig. 4.28(d).

### PROBLEMS

4.8. Find the product of inertia of the area shown about the  $x$  and  $y$  axes. The area is symmetrical by rotation about the origin.

4.9. Find the product of inertia of the area shown about the  $x$  and  $y$  axes.

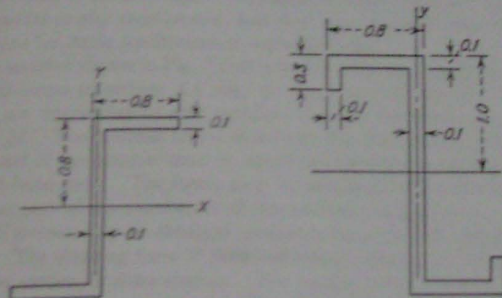


FIGURE 4.8

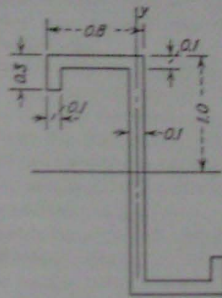


FIGURE 4.9

4.10. Find the product of inertia of the area shown about horizontal and vertical axes through the centroid.

4.11. Find the moments of inertia about the principal axes. Find the angles to the principal axes, and show these angles on a sketch of the area.

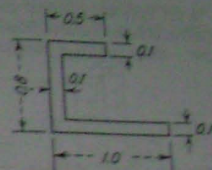


FIGURE 4.10

4.12. Find the moments of inertia about the principal axes. Show the angles to the principal axes on a sketch of the area.

4.13. Find the moments of inertia about principal axes through the centroid. Show the angles to the principal axes on a sketch of the area.

4.14. The effective area of the wing cross section shown has the following properties about the  $x$  and  $y$  axes through the centroid:  $I_x = 480 \text{ in.}^4$ ,  $I_y = 1620 \text{ in.}^4$ ,  $I_{xy} = 180 \text{ in.}^4$ . Find the principal axes and the moments of inertia about the principal axes.

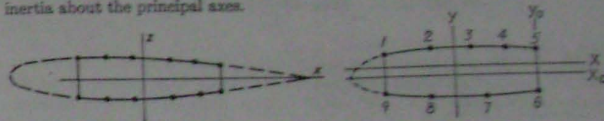


FIGURE 4.14

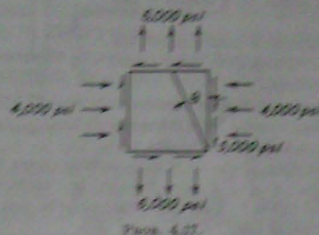
4.15. Find the principal axes and the moments of inertia about these axes if the wing cross section has the following properties:  $I_x = 420 \text{ in.}^4$ ,  $I_y = 1,280 \text{ in.}^4$ ,  $I_{xy} = -220 \text{ in.}^4$ .

4.16. The wing cross section shown is made up of elementary areas concentrated at nine points. The areas of the elements, and the coordinates of the areas with respect to reference axes  $x_0$  and  $y_0$  are tabulated below. Find the moments and product of inertia about the  $x_0$  and  $y_0$  axes. Find the location of the centroidal axes  $x$  and  $y$  and the moments and product of inertia about these axes; then find the principal axes through the centroid and the moments of inertia about the principal axes.

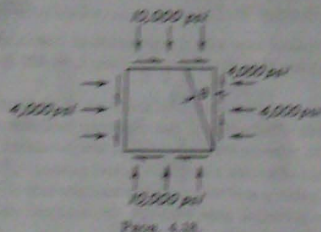
TABLE 4.6

| Element | Area  | $x_0$ | $y_0$ |
|---------|-------|-------|-------|
| 1       | 0.422 | -35.1 | 5.13  |
| 2       | 0.382 | -28.0 | 5.25  |
| 3       | 0.382 | -20.5 | 4.75  |
| 4       | 0.382 | -10.5 | 4.25  |
| 5       | 0.562 | 0     | 3.55  |
| 6       | 0.487 | 0     | -1.44 |
| 7       | 0.503 | -12.4 | -2.48 |
| 8       | 0.503 | -23.2 | -3.36 |
| 9       | 0.416 | -35.1 | -3.35 |

4.27. Find the principal stresses and principal planes for the element shown. Find stresses on inclined planes for values of  $\theta$  at intervals of  $15^\circ$  between  $0$  and  $180^\circ$ . Show results on sketches similar to those in Fig. 4.28.



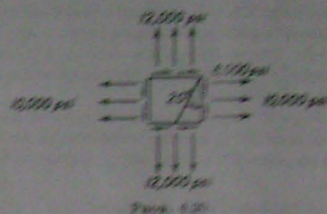
4.28. Find the principal stresses and principal planes for the element shown. Find the planes of maximum shear and the stresses on these planes. Find the stresses on the planes for  $\theta = 30^\circ$  and  $\theta = 120^\circ$ . Show all results on sketches.



4.29. An element on the upper surface of a wing carries compressive stresses of 30,000 psi and shearing stresses of 10,000 psi as shown. Find the principal planes and principal stresses. Find the stresses on a plane inclined  $30^\circ$  as shown.



4.30. Find the principal stresses and principal planes for the element shown. Find the stresses on the plane inclined  $20^\circ$ , as shown.



#### REFERENCE FOR CHAPTER 4

1. MAIR, J.: "Strength of Materials," Chap. VI, The Macmillan Company, New York, 1948.



## CHAPTER 5

## SHEAR AND BENDING-MOMENT DIAGRAMS

**5.1. Shears and Bending Moments.** In the design of all structural members except two-force members it is necessary to obtain the shearing forces and the bending moments at various cross sections of the members. When all the loads acting on a member are known, the resultant internal forces at any cross section may readily be obtained from the static

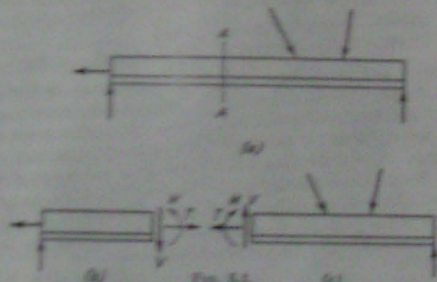


FIG. 5.1

equilibrium of the part of the member on either side of the cross section. For members which are loaded with coplanar forces, there are three force components at any cross section, and they may be found from the three equations for static equilibrium of coplanar forces.

The member shown in Fig. 5.1(a) is loaded with coplanar forces. The internal forces at section A-A may be represented by forces  $T$  and  $V$ , which are perpendicular and parallel to the cross section, and by a couple  $M$ . The internal forces at section A-A which act on the right-hand part of the member must be equal and opposite to those acting on the left-hand part. The forces may be obtained from the equilibrium of either part of the member, if all the external loads are known. The internal stresses may be obtained separately for each force and superimposed. The shearing force  $V$  produces stresses parallel to the plane of the cross section, or shear stresses. The tension force  $T$  and the bending moment  $M$  produce stresses normal to the cross section. In order for the tension force to produce stresses which are uniformly distributed

over the cross-sectional area, it must act at the centroid of the area, as shown in Fig. 5.2, since the centroid of an area was defined as the position of the resultant of parallel forces uniformly distributed over the area. The resultant tension force is therefore considered as acting through the centroid of the cross-sectional area when computing bending moments. The bending moment  $M$  is usually computed by taking moments of the external forces to the left of the cross section about the centroid of the cross section, since the forces  $V$  and  $T$  have no moment about this point.

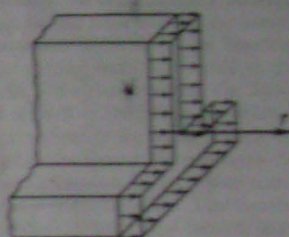


FIG. 5.2

The values of  $V$  and  $M$  are usually plotted for all points along the length of the member, and the resulting curves are called the shear and bending-moment diagrams. The positive direction for the shear in a horizontal beam is shown in Figs. 5.1(b) and (c), in which the resultant of the external forces to the left of the cross section is up, and the resultant to the right of the cross section is down. A positive bending moment produces compressive stress on the upper side of the beam. In the case of vertical beams there is no established sign convention, and the positive direction should be designated on each diagram. A different sign convention is occasionally used for the shear in a full-cantilever airplane wing, where uploads are assumed to produce positive shear. According to the normal sign convention the uploads on the left wing, as viewed looking forward at the airplane, would produce positive shear, but uploads on the right wing would produce negative shear.

**Example.** Plot the shear and bending-moment diagrams for the beam shown in Fig. 5.3(a).

**Solution.** The reactions  $R_1$  and  $R_2$  are first obtained by considering the entire beam as a free body.

$$\Sigma M_{A_1} = 100 \times 100 \times 10 - 20,000 \times 70 + 100R_2 = 0$$

$$R_2 = 13,000 \text{ lb}$$

$$\Sigma M_{A_2} = 100 \times 100 \times 110 + 20,000 \times 30 - 100R_1 = 0$$

$$R_1 = 17,000 \text{ lb}$$

$$\text{Check: } \Sigma F_x = 100 \times 100 - 17,000 + 20,000 - 13,000 = 0$$

The shear and bending moment for any cross section between the left end of the beam and the left reaction are found by considering the free body shown in Fig. 5.4.

$$V = 100x$$

$$M = 50x^2$$

The equation for  $V$  represents a straight line from the point  $x = 0$ ,  $V = 0$ , to the point  $x = 50$ ,  $V = 5,000$ , and is plotted in Fig. 5.3(b). The equation for  $M$  represents a parabola which is concave upward, and which passes through the

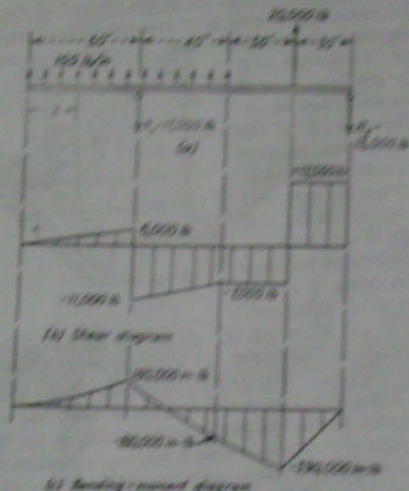


FIG. 5.3.

points  $x = 0$ ,  $M = 0$ , and  $x = 50$ ,  $M = 125,000$ , as shown in Fig. 5.3(c).

The portion of the beam under the distributed load, and to the right of the reaction, is next considered, as shown in Fig. 5.4.

$$V = 100x - 17,000$$

$$M = 50x^2 - 17,000x + 90$$

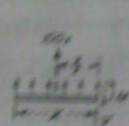


FIG. 5.4.

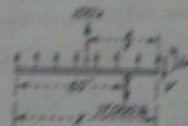


FIG. 5.5.

The equation for  $V$  is a straight line from the point  $x = 0$ ,  $V = -17,000$ , to the point  $x = 100$ ,  $V = -7,000$ , as shown in Fig. 5.3(b). The equation for  $M$  is a parabola which is concave upward and which passes through the points  $x = 0$ ,  $M = 125,000$ , and  $x = 100$ ,  $M = 180,000$ .

The portion of the beam shown in Fig. 5.6 is now considered.

$$V = 10,000 - 17,000 = -7,000$$

$$M = 10,000x - 50 - 17,000(x - 50)$$

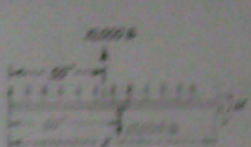


FIG. 5.6.

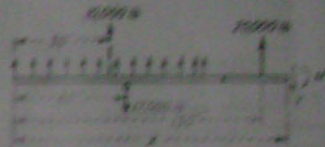


FIG. 5.7.

The equation for  $M$  represents a straight line through the points  $x = 100$ ,  $M = -180,000$ , and  $x = 150$ ,  $M = -200,000$ .

The portion of the beam shown in Fig. 5.7 is next considered.

$$V = 10,000 - 17,000 + 20,000 = 13,000$$

$$M = 10,000x - 50 - 17,000(x - 100) + 20,000(x - 150)$$

The equations for  $V$  and  $M$  represent straight lines, as shown in Fig. 5.3. A numerical check is obtained from the point  $x = 150$ , since the shear is equal to the right reaction and the bending moment is zero. Any other point may be checked by considering the part of the beam to the right of the point.

It is not necessary to write the equations for the shear and bending-moment diagrams in most cases. The numerical values shown in Figs. 5.3(b) and (c) may be calculated directly by considering the equilibrium of the beam to the left of the points. The shape of the diagrams between these points may then be sketched accurately by making use of the relations developed in the following article.

## 5.2. Relations between Load, Shear, and Bending Moment.

If a small element of a beam of length  $dx$ , as shown in Fig. 5.8, is considered as a free body, some useful relations may be developed. The beam carries a distributed load of  $w$  per unit length, and the load on the element is  $w dx$ , positive upward. From a summation of vertical forces on the element, the following equations are obtained:

$$V + w dx - (V + dV) = 0$$

or

$$\frac{dV}{dx} = w \quad (5.1)$$

Another relation is derived by a summation of moments about the right-hand cross section.

$$M + V dx + w dx \frac{dx}{2} - (M + dM) = 0$$



Simplifying and dropping the second-order differential,

$$\frac{dM}{dx} = V \quad (5.2)$$

Equation 5.1 shows that the load intensity at any point is equal to the derivative of the shear equation at that point. This relationship is seen to apply to the equations for  $V$  which were obtained in the Example problem of Art. 5.1. A differentiation of the equations for  $V$  gives values of  $dV/dx = 100$  for all points under the distributed load of  $w = 100$  lb/in., and  $dV/dx = 0$  for points where the beam is not loaded. A study of the shear diagram shown in Fig. 5.3(b) shows that the shear increases at a rate of 100 lb/in. under the distributed load and remains constant at unloaded sections of the beam. The concentrated loads or reactions are assumed to be distributed over a zero length of the beam, and the load intensity  $w$  is infinite at these points. The shear diagram will always have discontinuities at these points.

Equation 5.2 shows that the shear is equal to the derivative of the bending-moment equation at any point. This relationship is very useful in obtaining the shape of the bending-moment diagram between plotted points. A consideration of the bending-moment diagram of Fig. 5.3(c) shows that the moment curve must have a horizontal tangent at the left end of the beam, since the shear is zero at this point. The slope of the bending-moment curve must increase gradually from the left end of the beam to the left reaction, since the shear increases in this region. Just to the left of the reaction the bending moment is increasing at the rate of 6,000 in-lb/in., since the slope must be equal to the shear at this point. The slope of the bending-moment diagram just to the right of the left reaction is -11,000 in-lb/in. The bending-moment diagram has a negative slope for the entire region in which the shear is negative. For the region in which the shear is constant, the bending-moment curve must be a straight line, with a slope equal to the value of the shear.

All maximum or minimum values of the bending moment must occur where the shear is zero, since the derivative of the bending-moment equation must be zero. When the maximum bending moment occurs at a point under a distributed load, it is convenient to locate the point of the maximum moment from the shear diagram.

**5.3. Bending Moment as Area of Shear Diagram.** Equation 5.2 may be integrated between any two points  $A$  and  $B$  of a beam.

$$\int_A^B dM = \int_A^B V dx$$

$$M_B - M_A = \int_A^B V dx \quad (5.3)$$

or

The integral of Eq. 5.3 represents the area of the shear diagram between points  $A$  and  $B$ . The increase in the bending moment from point  $A$  to point  $B$  is therefore equal to the area of the shear diagram between these points. When the shear is negative, the area must be considered as negative. This principle is quite useful in obtaining or checking values of the bending moment. This method could be applied in the Example problem of Art. 5.1. The area of the shear curve of Fig. 5.3(b) between  $x = 0$ , and  $x = 60$ , is  $6,000 \times 60/2 = 180,000$  in-lb, which is the change in bending moment for this section of the beam. Since the bending moment is zero at the left end of the beam, this represents the bending moment at  $x = 60$ . Similarly, the area of the shear curve between points  $x = 60$  and  $x = 100$  is  $-360,000$  in-lb. The bending moment at  $x = 100$  is therefore  $180,000 - 360,000$ , or  $-180,000$  in-lb. For a beam such as this, where the bending moment is zero at each end, the total area of the shear diagram must be zero.

For beams carrying eccentric axial loads or couple loads, Eq. 5.3 does not apply. In deriving Eq. 5.3, the bending moment is assumed to be a continuous function between points  $A$  and  $B$ . For the beam shown in Fig. 5.9, the bending-moment curve has a discontinuity at the point where the couple is applied. The area under the shear diagram to the left of a point near the right support is obviously not equal to the bending moment for the point. The total area of the shear diagram is not zero, even though the bending moment is zero at each end of the beam. Equation 5.3 can, however, be used for parts of the beam if the discontinuity of the bending-moment curve is not between the points  $A$  and  $B$ .

For beams carrying distributed loads, the shear may be obtained as the area under the load curve. By integrating Eq. 5.1,

$$\int_A^B dV = \int_A^B w dx$$

or

$$V_B - V_A = \int_A^B w dx \quad (5.4)$$

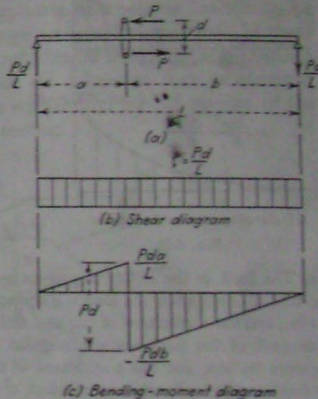


FIG. 5.9.



The integral of Eq. 5.4 represents the area under the load diagram between points  $A$  and  $B$ . This area is equal to the increase in shear between these points.

**Example 1.** Find the equations of the shear and bending-moment diagrams for the beam shown in Fig. 5.10(a) by the following three methods:

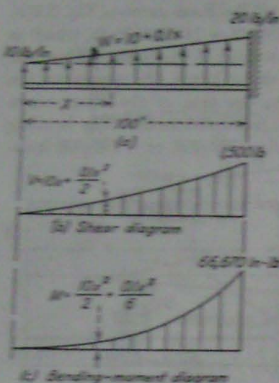


FIG. 5.10.

- By considering forces to the left of a section, as in Art. 5.1.
- By integration of the equations for the load and shear curves.
- By obtaining the areas of the load and shear curves geometrically.

**Solution.** The load intensity increases from 10 lb/in. when  $x = 0$  to 30 lb/in. when  $x = 100$ . The load at any point  $x$  is therefore represented by the equation

$$w = 10 + 0.1x \quad (5.5)$$

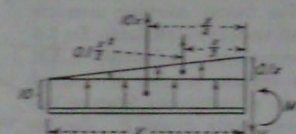


FIG. 5.11.

a. The load to the left of any cross section may be considered as shown in Fig. 5.11. One part of the load is uniformly distributed at 10 lb/in. for a length of  $x/3$  in., and has a resultant of  $10x/3$  at a distance  $x/3$  from the cross section. The other part of the load has a triangular distribution with a value of  $0.1x$  at the cross section, and has a resultant of  $0.1x^2/6$  acting at a distance  $x/3$  from the cross section. From a summation of vertical forces,

$$V = 10x + 0.1 \frac{x^2}{2}$$

From a summation of moments about the cross section,

$$M = 10x \frac{x}{3} + 0.1 \frac{x^2}{2} \frac{x}{3}$$

$$M = 10 \frac{x^2}{2} + 0.1 \frac{x^3}{6}$$

The diagrams for  $V$  and  $M$  are plotted in Fig. 5.10.

b. The equations for  $V$  and  $M$  may be obtained by direct integration of the equation for the load  $w$ . From Eqs. 5.4 and 5.5,

$$V - 0 = \int_0^x (10 + 0.1x) dx$$

$$V = 10x + 0.1 \frac{x^2}{2}$$

From Eq. 5.3,

$$M - 0 = \int_0^x V dx$$

$$M = \int_0^x \left( 10x + 0.1 \frac{x^2}{2} \right) dx$$

$$M = 10 \frac{x^2}{2} + 0.1 \frac{x^3}{6}$$

or

These equations for  $V$  and  $M$  correspond with those previously obtained.

c. The shear  $V$  may be obtained as the area under the load curve to the left of the point, since the shear is zero at the left end of the beam. The area under the load curve is equal to the sum of the areas of the rectangle and the triangle shown in Fig. 5.11.

$$V = 10x + 0.1 \frac{x^2}{2}$$

The shear curve may be plotted in two parts as shown in Fig. 5.12. The shear resulting from the uniformly distributed load is represented by the triangular diagram having the value  $10x$  at the point  $x$ . The shear resulting from the triangular load is represented by the parabola having a value of  $0.1x^2/2$  at the point  $x$ . Since the bending moment at the left end of the beam is zero, the bending moment at point  $x$  is equal to the area under the shear curve to the left of the point. The area of the triangle shown in Fig. 5.12 is  $10x^2/2$ . The area of the parabola is one-third of the product of the base and the maximum ordinate, or  $(0.1x^2/2)(x/3)$ . The total bending moment is the sum of the areas,

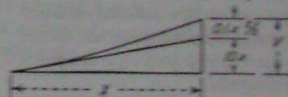


FIG. 5.12.

$$M = 10 \frac{x^2}{2} + 0.1 \frac{x^3}{6}$$

This checks the equation previously obtained.

**Example 2.** The aerodynamic loads on an airplane wing cannot be represented by a simple equation. The load per inch of span,  $w$ , for the airplane wing shown in Fig. 5.13(a) is tabulated in column (2) of Table 5.1 for various points along the span. Find the shear and bending-moment diagrams for the wing.

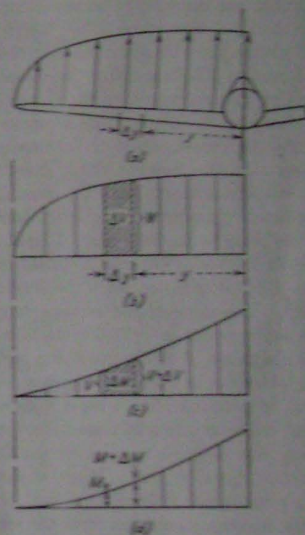


FIG. 5.13.



The integral of Eq. 5.4 represents the area under the load diagram between points *A* and *B*. This area is equal to the increase in shear between these points.

**Example 1.** Find the equations of the shear and bending-moment diagrams for the beam shown in Fig. 5.10(a) by the following three methods:

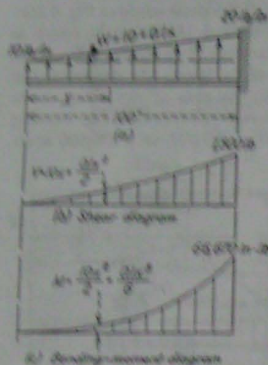


FIG. 5.10.

a. The load to the left of any cross section may be considered as shown in Fig. 5.11. One part of the load is uniformly distributed at 10 lb/in. for a length of  $x$  in., and has a resultant of  $10x$  at a distance  $x/2$  from the cross section. The other part of the load has a triangular distribution with a value of  $0.1x$  at the cross section, and has a resultant of  $0.1x^2/2$  acting at a distance  $x/3$  from the cross section. From a summation of vertical forces,

$$V = 10x + 0.1 \frac{x^2}{2}$$

From a summation of moments about the cross section,

$$M = 10x \frac{x}{2} + 0.1 \frac{x^3}{6}$$

or

$$M = 10 \frac{x^2}{2} + 0.1 \frac{x^3}{6}$$

The diagrams for  $V$  and  $M$  are plotted in Fig. 5.10.

b. The equations for  $V$  and  $M$  may be obtained by direct integration of the equation for the load  $w$ . From Eqs. 5.4 and 5.5,

$$V = 0 = \int_0^x (10 + 0.1x) dx$$

a. By considering forces to the left of a section, as in Art. 5.1.

b. By integration of the equations for the load and shear curves.

c. By obtaining the areas of the load and shear curves geometrically.

**Solution.** The load intensity increases from 10 lb/in. when  $x = 0$  to 20 lb/in. when  $x = 100$ . The load at any point  $x$  is therefore represented by the equation

$$w = 10 + 0.1x \quad (5.3)$$

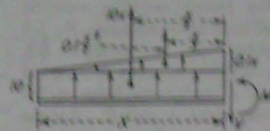


FIG. 5.11.

or

$$V = 10x + 0.1 \frac{x^2}{2}$$

From Eq. 5.3,

$$M = 0 = \int_0^x V dx$$

$$M = \int_0^x (10x + 0.1 \frac{x^2}{2}) dx$$

or

$$M = 10 \frac{x^2}{2} + 0.1 \frac{x^3}{6}$$

These equations for  $V$  and  $M$  correspond with those previously obtained.

c. The shear  $V$  may be obtained as the area under the load curve to the left of the point, since the shear is zero at the left end of the beam. The area under the load curve is equal to the sum of the areas of the rectangle and the triangle shown in Fig. 5.11.

$$V = 10x + 0.1 \frac{x^2}{2}$$

The shear curve may be plotted in two parts as shown in Fig. 5.12. The shear resulting from the uniformly distributed load is represented by the triangular diagram having the value  $10x$  at the point  $x$ . The shear resulting from the triangular load is represented by the parabola having a value of  $0.1x^2/2$  at the point  $x$ . Since the bending moment at the left end of the beam is zero, the bending moment at point  $x$  is equal to the area under the shear curve to the left of the point. The area of the triangle shown in Fig. 5.12 is  $10x^2/2$ . The area of the parabola is one-third of the product of the base and the maximum ordinate, or  $(0.1x^2/2)(x/3)$ . The total bending moment is the sum of the areas,

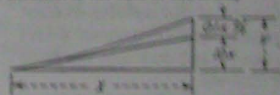


FIG. 5.12.

$$M = 10 \frac{x^2}{2} + 0.1 \frac{x^3}{6}$$

This checks the equation previously obtained.

**Example 2.** The aerodynamic loads on an airplane wing cannot be represented by a simple equation. The load per inch of span,  $w$ , for the airplane wing shown in Fig. 5.13(a) is tabulated in column (2) of Table 5.1 for various points along the span. Find the shear and bending-moment diagrams for the wing.

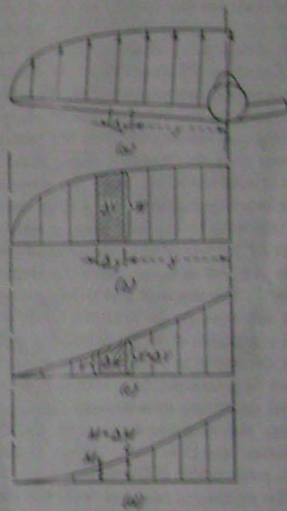


FIG. 5.13.

**Solution.** The values of the shear and bending moment at various points along the wing are tabulated in Table 11. The points are called stations and

TABLE 11

| No.                     | Load intensity                   | Dist. between stations | Shear constant | Shear        | Moment increment       | Bending moment  |
|-------------------------|----------------------------------|------------------------|----------------|--------------|------------------------|-----------------|
| Dist. from station line | From left load, rectangular load | From station line (1)  | $w$ , lb       | $V$ , lb     | $M$ , lb-ft            | $M$ , lb-ft     |
| 1, ft. (2)              | $x$ , ft. (3)                    | $x_1$ , ft. (4)        | $w$ , lb (5)   | $V$ , lb (6) | $\Delta M$ , lb-ft (7) | $M$ , lb-ft (8) |
| 0                       | 0                                | 0                      | 0              | 0            | 0                      | 0               |
| 10                      | 10                               | 10                     | 100            | 40           | 200                    | 200             |
| 20                      | 20                               | 20                     | 100            | 0            | 1,100                  | 1,300           |
| 30                      | 30                               | 30                     | 1,000          | 1,000        | 30,300                 | 31,600          |
| 40                      | 40                               | 40                     | 1,100          | 2,100        | 44,300                 | 75,900          |
| 50                      | 50                               | 50                     | 1,100          | 3,200        | 61,300                 | 137,200         |
| 60                      | 60                               | 60                     | 1,000          | 4,200        | 82,300                 | 219,500         |
| 70                      | 70                               | 70                     | 1,500          | 5,700        | 126,300                | 345,800         |
| 80                      | 80                               | 80                     | 2,000          | 7,700        | 197,300                | 543,100         |
| 90                      | 90                               | 90                     | 2,100          | 9,800        | 284,300                | 827,400         |
| 100                     | 100                              | 100                    | 2,000          | 11,800       | 399,300                | 1,226,700       |
| 110                     | 110                              | 110                    | 2,000          | 13,800       | 549,300                | 1,776,000       |
| 120                     | 120                              | 120                    | 2,400          | 16,200       | 749,300                | 2,525,300       |
| 130                     | 130                              | 130                    | 2,400          | 18,600       | 999,300                | 3,524,600       |

are designated by their distance  $x$  from the station line of the airplane, as shown in Fig. 3.13(a). These distances are measured along the wing rather than horizontally, since the air loads are perpendicular to the wing. The distances between the stations,  $\Delta x$ , are tabulated in column (5). The value of the shear at any point is obtained as the area under the load curve to the left of the point.

The load curve is assumed to be a series of straight lines between the given points, as shown in Fig. 3.13(b), and the area is obtained as the area of the series of its triangles. The areas of the triangles are obtained as the product of the average height  $w$ , and the base  $\Delta x$ , and are tabulated in column (6). The change in shear  $\Delta V$  between two stations is equal to the area of the load curve between the stations, as shown in Fig. 3.13(b) and (c). The shear  $V$  is then obtained in column (7) by summation of the areas in column (6). The change in bending moment between two stations is equal to the area under the shear curve. This area is also assumed triangular and is obtained by multiplying the area of the shear at the adjacent stations by one-half the distance between the stations. The areas of these triangles are tabulated in column (8). The bending moments are obtained in column (9) by a summation of the areas in column (8).

**3.4 Members Resisting Noncoplanar Loads.** The resultant forces on any cross section of a member resisting noncoplanar loads may be represented by six components. These are usually considered as three force components along mutually perpendicular axes and three moment components about these axes. These six unknowns may be obtained from the six equations for the static equilibrium for the portion of the structure on either side of the cross section.



FIG. 3.14

The forces and moments acting at a cross section of an airplane wing are shown in Fig. 3.14. The  $x$  and  $z$  axes are in the plane of the cross section, and the forces along these axes are the horizontal and vertical shearing forces on the cross section. The  $y$  axis should be through the centroid of the cross-sectional area if the stresses resulting from the tension force  $T$  are uniformly distributed over the area. In the case of an airplane wing, the tension force  $T$  is negligible, and the  $x$  and  $z$  axes are selected arbitrarily before calculating the position of the centroid of the area. The couple  $M_x$  and  $M_z$  represent wing bending moments resulting from vertical and horizontal loads. The couple  $M_y$  represents the wing torque. Shear and bending-moment diagrams are obtained separately for the loads in the  $yz$  plane and for loads in the  $xz$  plane.



**Example.** Plot the shear and bending-moment diagrams for the landing-gear oleo strut shown in Figs. 2.11 and 2.12.

**Solution.** The shears and bending moments for loads in the  $VS$  plane are computed separately from the shears and bending moments for loads in the  $VD$  plane. Figure 5.15(a) shows the loads in the  $VS$  plane. The loads perpendicular to this plane do not affect the shears and bending moments in the plane, but are shown in the figure. The shear in the  $VS$  plane is produced

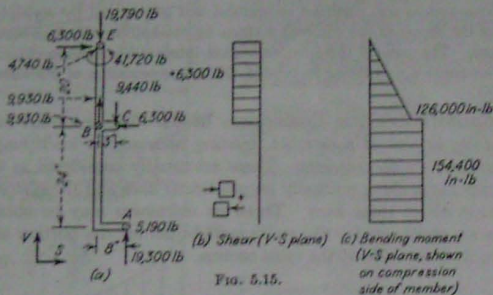


FIG. 5.15.

by components of the loads along the  $S$  axis, and is zero below point  $B$  and 6,300 lb above point  $B$ , as shown in Fig. 5.15(b). The positive direction of the shear is indicated on the shear diagram since there is no established convention for positive shear in a member of this type. The bending-moment diagram for loads in the  $VS$  plane is shown in Fig. 5.15(c). The bending moment just below point  $B$  is obtained by taking moments of all forces below this point.

$$M = 19,300 \times 8 = 154,400 \text{ in.-lb}$$

This bending moment is constant for the lower part of the member, since the 19,300-lb load has the same moment about any point on the center line of the member. The bending moment a small distance above point  $B$  is also obtained by taking moments of all forces below the point.

$$M = 19,300 \times 8 - 9,440 \times 3 = 126,000 \text{ in.-lb}$$

The bending moment above point  $B$  decreases linearly to zero at the top of the member, because the moment of the horizontal 6,300-lb force at  $B$  is subtracted from the moment of 126,000 in.-lb. The value at  $B$  may be checked by taking moments of the forces above the point.

$$M = 6,300 \times 20 = 126,000 \text{ in.-lb}$$

The bending-moment diagram is shown on the compression side of the member, and all values on the diagram are shown as positive.

It is obvious that the bending moment at any point cannot be obtained as the area under the shear diagram below the point. The vertical loads at  $A$  and  $C$

produce discontinuities in the bending-moment diagram, and extreme care must be used in obtaining bending moments from the shear areas in such cases. The slope of the bending-moment curve is equal to the shear, however, and the change in bending moment between two points is equal to the shear area between the points if there is no discontinuity between the points.

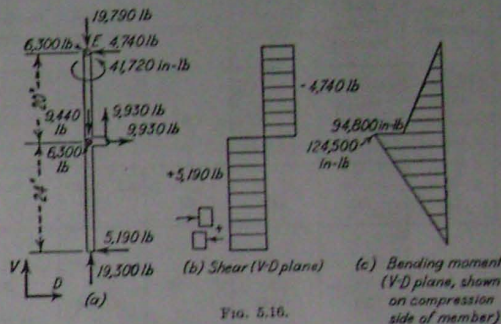
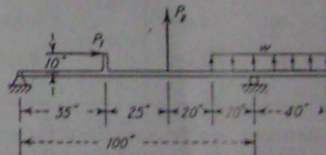


FIG. 5.16.

The shear and bending-moment diagrams for loads in the  $VD$  plane are shown in Fig. 5.16. The shear diagram is obtained from the components of forces along the  $D$  axis, and is shown in Fig. 5.16(b). The bending-moment diagram shown in Fig. 5.16(c) is obtained from the moments of  $V$  and  $D$  components of forces. The bending moments are obtained by taking moments of the forces below the cross section and are checked by taking moments of the forces above the cross section, as in the previous calculations. The bending-moment diagram consists of straight lines because the shear is constant. It is necessary to compute values of the bending moment just above and just below point  $C$  and also at the ends of the member.

### PROBLEMS

5.1. Plot the shear and bending-moment diagrams for the beam shown if  $P_1 = 0$ ,  $P_2 = 400$  lb, and  $w = 10$  lb/in.

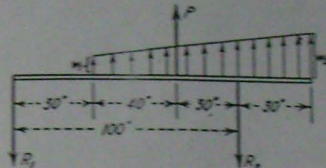


FIGS. 5.1 to 5.4.

5.2. Plot the shear and bending-moment diagrams for the beam shown if  $P_1 = 0$ ,  $P_2 = 1,000$  lb, and  $w = 5$  lb/in.



- 5.3. Plot the shear and bending-moment diagrams for the beam shown if  $P_1 = 1,000$  lb,  $P_2 = 400$  lb, and  $w = 0$ .
- 5.4. Plot the shear and bending-moment diagrams for the beam shown if  $P_1 = 2,000$  lb,  $P_2 = 100$  lb, and  $w = 10$  lb/in.
- 5.5. Plot the shear and bending-moment diagrams for the beam shown if  $P = 0$ ,  $w_1 = 0$ , and  $w_2 = 10$  lb/in.



FIGURES 5.5 AND 5.6.

- 5.6. Plot the shear and bending-moment diagrams for the beam shown if  $P = 2,000$  lb,  $w_1 = 10$  lb/in., and  $w_2 = 20$  lb/in.
- 5.7. Calculate the shears and bending moments at 20-in. intervals for the beam of Fig. 5.10(c), using the tabular method of Example 2, Art. 5.3. Find the percentage error for each value of the bending moment.
- 5.8. Repeat Prob. 5.7, obtaining values at 10-in. intervals.
- 5.9. Plot the bending-moment diagrams for the landing-gear member shown in Figs. 2.12(c) and (d).
- 5.10. Plot the bending-moment diagrams for the landing-gear oleo strut shown in Fig. 2.11, if the loads applied to the axle are 20,000 lb in the  $V$  direction and 6,000 lb in the  $D$  direction.

## CHAPTER 6

## SHEAR AND BENDING STRESSES IN SYMMETRICAL BEAMS

**6.1. Flexure Formula.** After finding the values of the shear and bending moment at a cross section of a beam, it is necessary to find the unit stresses on the cross section in order to design the beam. The unit stress is the force intensity at any point, and has units of force per unit area, or pounds per square inch (psi) in common engineering units. Where the term *stress* is used in future discussion, it will be assumed to mean unit stress. In this chapter, it will be assumed that the beam cross section is symmetrical about either a horizontal or a vertical axis through its centroid, so that these axes will be principal axes. The stresses resulting from axial forces, shearing forces, and bending moments will be computed separately and superimposed.

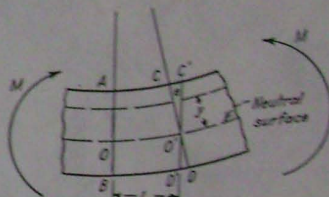


FIG. 6.1.

An initially straight beam is deflected to a circular arc when loaded in pure bending as shown in Fig. 6.1. It has been found experimentally that a plane cross section of the beam remains plane after bending. The cross sections  $AB$  and  $CD$  shown in Fig. 6.1 would both be vertical planes before bending, and would become radial planes after bending. If the beam is considered as made up of longitudinal "fibers," the fibers in the upper part of the beam will be compressed and those in the lower part will be stretched. The fibers on some intermediate surface, called the *neutral surface*, will remain the same length. The intersection of the neutral surface and any cross section is called the neutral axis for the cross section. If cross sections  $AB$  and  $CD$  are originally a unit distance apart, the unit elongation  $e$  of any fiber between these cross sections is